UNIVERSIDAD NACIONAL DE INGENIERÍA

FACULTAD DE CIENCIAS

TESIS

"A STUDY OF DIMENSIONAL AND RECURRENCE PROPERTIES OF INVARIANT MEASURES OF FULL-SHIFT AND AXIOM A SYSTEMS: GENERIC BEHAVIOUR"

PARA OBTENER EL GRADO ACADÉMICO DE DOCTOR EN CIENCIAS CON MENCIÓN EN MATEMÁTICA

ELABORADA POR:

ALEXANDER PAUL CONDORI HUAMÁN

ASESOR:

Dr. ROGER JAVIER METZGER ALVÁN

ASESOR EXTERNO:

Dr. SILAS LUIZ DE CARVALHO

LIMA - PERÚ

2019

Dedicado a mis padres:

Ayde y César.

Acknowledgments

To God, for giving me the opportunity to conclude this phase of my life.

To my parents, Ayde and César, for their love, affection and unconditional support.

To all my family, for the help and the incentive that they have always given me. I always carry them in my heart.

To my advisers Roger J. Metzger Alván and Silas Luiz de Carvalho, for the patience, the disposition and the trust placed in me.

To the prized professors Martha Nina, Enrique Avilés, Raúl Achallma, José Luis Condori, Rocio, Willy, among others who were my undergraduate mentors. Also, to the prized professors Enoch Humberto, Arturo Fernandez, Gonzalo Panizo, Philippe Thieullen, among others who were my mentors during the doctorate course, and in special to my adviser Roger J. Metzger Alván and my co-adviser Silas luiz de Carvalho of Instituto de Ciências Exatas (ICEX) of Universidade Federal de Minas Gerais, for their guidance and teachings.

To my friends and colleagues at the Instituto de Matemáticas y Ciencias Afines (IMCA), for friendship and companionship.

To CONCYTEC-FONDECYT through Fondecyt C.G. 176-2015, for the doctorate scholarship.

Contents

Abstract

In this thesis, we are interested in characterizing typical (generic) dimensional properties of invariant measures associated with the full-shift system, T, in a product space whose alphabet is uncountable. More specifically, we show that the set of invariant measures with upper Hausdorff dimension equal to zero and lower packing dimension equal to infinity is a dense G_δ subset of $\mathcal{M}(T)$, the space of T-invariant measures endowed with the weak topology. We also show that the set of invariant measures with upper rate of recurrence equal to infinity and lower rate of recurrence equal to zero is a dense G_δ subset of $\mathcal{M}(T)$. Furthermore, we show that the set of invariant measures with upper quantitative waiting time indicator equal to infinity and lower quantitative waiting time indicator equal to zero is also residual in $\mathcal{M}(T)$.

For topological dynamical systems with a dense set (in the weak topology) of periodic measures, we show that a typical invariant measure has, for each $q > 0$, zero lower q-generalized fractal dimension. This implies, in particular, that a typical invariant measure has zero upper Hausdorff dimension and zero lower rate of recurrence. Of special interest is the full-shift system (X, T) (where $X = M^{\mathbb{Z}}$ is endowed with a sub-exponential metric and the alphabet M is a perfect and compact metric space), for which we show that a typical invariant measure has, for each $q > 1$, infinite upper q-correlation dimension. Under the same conditions, we show that a typical invariant measure has, for each $s \in (0, 1)$ and each $q > 1$, zero lower s-generalized and infinite upper q-generalized dimensions.

Finally, for measure preserving dynamical systems on metric spaces, we present sufficient conditions involving the upper and lower pointwise dimensions of the measure in order to obtain upper and lower bounds for its generalized fractal dimensions. We also obtain an extension of Young's Theorem [59] involving the generalized fractal dimensions of the Bowen-Margulis measure of an Axiom A system. Furthermore, for Axiom A systems, we show that the set of invariant measures with zero correlation dimension, under a hyperbolic metric, is generic.

Resumen

En esta tesis, nos interesa caracterizar las propiedades dimensionales típicas (genéricas) de medidas invariantes asociadas al sistema de cambio completo (full-shift system), (X, T), en un espacio producto cuyo alfabeto es no numerable. Se muestra que el conjunto de medidas invariantes que tienen dimensión de Hausdorff superior cero y dimensión de empaquetamiento inferior infinita es un subconjunto G_δ-denso de $\mathcal{M}(T)$, el espacio de medidas T-invariantes dotadas con la topología débil. También se muestra que el conjunto de medidas invariantes con tasa de recurrencia superior igual a infinito e inferior igual a cero es un subconjunto G_δ-denso de $\mathcal{M}(T)$. Además, se muestra que el conjunto de medidas invariantes con un indicador de tiempo de espera cuantitativo superior infinito e inferior cero es residual en $\mathcal{M}(T)$.

Para sistemas dinámicos topológicos con un conjunto denso de medidas periódicas, se muestra que una medida invariante típica tiene, para cada $q > 0$, q-dimensión fractal generalizada inferior igual a cero. Esto implica, en particular, que una medida invariante típica tiene dimensión de Hausdorff superior y tasa de recurrencia inferior iguales a cero. De especial interés es el sistema de cambio completo (full-shift system), (X, T), (donde $X = M^{\mathbb{Z}}$ es dotado de una métrica sub-exponencial y M es un espacio métrico perfecto y compacto), para el cual se muestra que una medida invariante típica tiene, para cada $q > 1$, q-dimensión de correlación superior infinita. Bajo las mismas condiciones, una medida invariante típica tiene, para cada $s \in (0, 1)$ y cada $q > 1$, dimensión inferior s-generalizada igual a cero y dimensión superior q-generalizada infinita.

Finalmente, para sistemas dinámicos que preservan medidas sobre espacios métricos, presentamos condiciones suficientes que involucran las dimensiones puntuales superior e inferior de la medida para obtener límites superiores e inferiores para sus dimensiones fractales generalizadas. También se obtiene una extensión del Teorema de Young que involucra las dimensiones fractales generalizadas de la medida de Bowen-Margulis de un sistema Axioma A. Además, para sistemas Axioma A, se muestra que una medida invariante típica tiene dimensión de correlación cero, bajo una métrica hiperbólica.

INTRODUCTION

Background and State of Art

"The dimension of invariant sets is among the most important characteristics of dynamical systems", as mentioned by Pesin in [42]. Under this premise, the importance of studying invariant sets of dynamical systems is clear and the dimension theory undoubtedly points to this. In this way, this thesis focuses on some dimensional properties of invariant measures associated with some specific dynamical systems. The information obtained by these dimensions are used in the characterization of the invariant sets where these measures are supported.

The classical and intuitive idea of dimension is perhaps what refers to an entire dimension suitable for Euclidean spaces, known as the typological dimension. This idea can be traced back, at least, to Poincaré, but it was only around 1922 that Urysohn and Menger formalised this notion. Naturally, the dimension of a point is zero; the dimension of a line is 1; the dimension of a plane is 2; the dimension of $\mathbb{R}^d$ is d (see Definition I.2). However, the intuitive idea of topological dimension seems to be insufficient when one tries to determine the dimension of certain sets of rather exotic structure which naturally rise, for example, in the theory of dynamical systems.

Soon after the discovery of such exotic structures, the so-called strange attractors, they became the focus of attention for many researchers who, among other problems, tried to obtain relations between the dimensions of these attractors and other invariants of the dynamical system, such as Lyapunov exponents and entropy (which characterize instability and stochasticity). Classic examples of these are the Lorenz attractor, the

Smale–Williams solenoid and Smale horseshoe. The latter is an example of a hyperbolic invariant set whose local topological structure is the product of different Cantor-like sets.

There are several different notions of dimension for more general sets, some easier to compute and others more convenient in applications. One of them is, and could be said to be the most popular of all, the Hausdorff dimension, introduced in 1919 by Hausdorff and also by Caratheodory, which gives a notion of size useful for distinguishing between sets of zero Lebesgue measure. This notion was widely investigated and widely used, among others, in the theory of dynamical systems, where many interesting invariant sets have zero Lebesgue measure, and later in function theory, mainly by Besicovitch and his school.

In 1975, Mandelbrot [33] has also observed the presence of these "strange" sets in some physical phenomena (as Julia and Mandelbrot sets). He called a set like these *fractal*, and defined it vaguely as the set whose Hausdorff dimension is strictly greater than its topological dimension. Mandelbrot also revealed an important aspect of the qualitative behavior of dynamical systems: assume that a physical system admits a group of scale similarities, i.e., that it "reproduces" itself on smaller scales. From a mathematical point of view, this means that the dynamical system, which describes the physical phenomenon, possesses invariant sets of a special self-similar structure.

The works of Hausdorff, Besicovitch and Mandelbrot gave shape to a new field in mathematics called *fractal geometry*. The results of this theory were widely used in different areas of science. Its importance lies in its independence of scaling. The rate of such scale is quantitatively characterized by a fractal dimension. The interaction of many individual fractals (often infinite), results in a multifractal, with a much more complicated topological structure.

Unfortunately, the Hausdorff dimension of relatively simple sets can be very hard to calculate; besides, the notion of Hausdorff dimension is not completely adapted to the dynamics per se (for instance, if Z is a periodic orbit, then its Hausdorff dimension is zero, regardless to whether the orbit is stable, unstable, or neutral). This fact led to the introduction of other characteristics with which it is possible to estimate the size of irregular sets. For this reason, some of these quantities were also branded as "dimensions" (although some of them lack some basic properties satisfied by Hausdorff dimension, such as σ-stability (see [34])). Several good candidates were proposed; among them are the correlation dimension, the information dimension, the box dimension, entropy dimension, etc.

From the notion of (Hausdorff) dimension of sets, one may propose the concept of dimension of a finite measure, namely, as the supreme one among the (Hausdorff) dimensions of the sets with total measure (that is, the upper Hausdorf dimension of a measure; see Definition I.7).

The importance of the dimension of an invariant measure, one might say, is due to the fact that, for a dynamical system, the fractal dimensions of an invariant measure provide more relevant information than the fractal dimensions of its invariant sets, or even the fractal dimensions of its topological support; the point is that invariant sets and topological supports usually contain superfluous sets (that is, measurable sets of zero measure). Thus, by establishing the fractal dimensions of invariant measures, one has a more precise information about the structure of the relevant sets (that is, the sets of positive measure) of a dynamical system (see [4, 40, 42] for a more detailed discussion).

Within the known literature, we may highlight the results obtained for hyperbolic ergodic measures. Eckmann and Ruelle, in [16], discussed the existence of the pointwise dimensions (see Definition I.8) of hyperbolic invariant measures (i.e, measures which are invariant under diffeomorphisms with non-zero Lyapunov exponents almost everywhere). This led to the problem of whether a hyperbolic ergodic measure is exact dimensional (i.e, whether the lower and upper local dimensions coincide almost everywhere; see Definition I.8). This problem was later known as the Eckmann-Ruelle conjecture, and has been recognized as one of the main problems in the interface between the dimension theory and the dynamical systems theory. Its importance in dimension theory of dynamical systems is compared to the importance of Shannon-McMillan-Breiman Theorem in Ergodic Theory.

In [59], Young showed that the hyperbolic measures which invariant under surface diffeomorphisms are exact dimensional, by providing a relation of the local dimensions involving the metric entropy and the Lyapunov exponents of the system. Later, Ledrappier

[29] extended this relation for general Sinai-Ruelle-Bowen measures. Finally, Eckmann-Ruelle conjecture was affirmatively answered by Barreira, Pesin and Schmeling in [5].

Hausdorff and Packing dimension of a measure and topological dimension of a set

The topological dimension of a set X is defined as follows. Let $\alpha = (A_i)_{i\in I}$ be a family of subsets of X indexed by a set I. For each $x \in X$, let

$$\mathrm{ord}_x(\alpha) := \mathrm{card}\{i \in I \mid x \in A_i\} - 1.$$

One says that the quantity $\mathrm{ord}_x(\alpha) \in \{-1\} \cup \mathbb{N} \cup \{\infty\}$ is the order of α at the point x. One defines the (global) order $\mathrm{ord}(\alpha) \in \{-1\} \cup \mathbb{N} \cup \{\infty\}$ of the family α by

$$\mathrm{ord}(\alpha) := \sup_{x\in X} \mathrm{ord}_x(\alpha).$$

Let $\alpha = (A_i)_{i\in I}$ and $\beta = (B_j)_{j\in J}$ be two covers of X. One says that β is finer than α if, for every $j \in J$, there exists $i \in I$ such that $B_j \subset A_i$.

Definition I.1. Let X be a topological space. Let $\alpha = (U_i)_{i\in I}$ be a finite open cover of X. One defines the quantity $D(\alpha)$ by

$$D(\alpha) := \min_{\beta} \mathrm{ord}(\beta),$$

where β runs over all finite open covers of X that are finer than α.

Definition I.2. Let X be a topological space. The topological dimension $\dim(X) \in \{-1\} \cup \mathbb{N} \cup \{\infty\}$ of X is given by $\dim(X) := \sup_\alpha D(\alpha)$ where α runs over all finite open covers of X.

In what follows, (X, d) is an arbitrary metric space and $\mathcal{B} = \mathcal{B}(X)$ is its Borel σ-algebra.

Definition I.3 (radius packing ϕ-premeasure, [14])**.** Let $\emptyset \neq E \subset X$, and let $0 < \delta < 1$. A δ-*packing* of E is a countable collection of disjoint closed balls $\{B(x_k, r_k)\}_k$ with centers

$x_k \in E$ and radii satisfying $0 < r_k \leq \delta/2$, for each $k \in \mathbb{N}$ (the centers x_k and radii r_k are considered part of the definition of the δ-packing). Given a measurable function ϕ, the radius packing (ϕ, δ)-premeasure of E is given by the law

$$P_\delta^\phi(E) = \sup \left\{ \sum_{k=1}^{\infty} \phi(2r_k) \mid \{B(x_k, r_k)\}_k \text{ is a } \delta\text{-packing of E} \right\}.$$

Letting $\delta \to 0$, one gets the so-called *radius packing ϕ-premeasure*

$$P_0^\phi(E) = \lim_{\delta \to 0} P_\delta^\phi(E).$$

One sets $P_\delta^\phi(\emptyset) = P_0^\phi(\emptyset) = 0$.

It is worth mentioning that, although open and closed balls in $\mathbb{R}^N$ possess nice regularity properties (for example, the diameter of a ball is twice its radius, and open and closed balls of the same radius have the same diameter), this may not be the case in arbitrary metric spaces. As it was observed by Cutler in [14], the possible absence of such regularity properties means that the usual measure construction based on diameters can lead to packing measures with undesirable features (see [14] for the details).

It is easy to see that P_0^ϕ is non-negative and monotone. Moreover, P_0^ϕ generally fails to be countably sub-additive. One can, however, build an outer measure from P_0^ϕ by applying Munroe's Method I construction, described both in [36] and [45]. This leads to the following definition.

Definition I.4 (radius packing ϕ-measure, [14])**.** The radius packing ϕ-measure of $E \subset X$ is defined to be

$$P^\phi(E) = \inf \left\{ \sum_k P_0^\phi(E_k) \mid E \subset \bigcup_k E_k \right\}. \tag{1}$$

The infimum in (1) is taken over all countable coverings $\{E_k\}_k$ of E. It follows that P is an outer measure on the subsets of X.

In an analogous fashion, one may define the Hausdorff ϕ-measure. The theory of

Hausdorff measures in general metric spaces is a well-explored topic; see, for example, the treatise by Rogers [45].

Definition I.5 (Hausdorff ϕ-measure, [14])**.** For $E \subset X$, the outer measure $H^\phi(E)$ is defined by

$$H^\phi(E) = \lim_{\delta \to 0} \inf \left\{ \sum_{k=1}^{\infty} \phi(\mathrm{diam}(E_k)) \mid \{E_k\}_k \text{ is a } \delta\text{-covering of } E \right\}, \tag{2}$$

where a δ-*covering* of E is any countable collection $\{E_k\}_k$ of subsets of X such that, for each $k \in \mathbb{N}$, $E \subset \bigcap_k E_k$ and $\mathrm{diam}(E_k) \leq \delta$. If no such δ-covering exists, one sets $H^\phi(E) = \inf \emptyset = \infty$.

Of special interest is the situation where given $\alpha > 0$, one sets $\phi(t) = t^\alpha$. In this case, one uses the notation P_0^α, and refers to $P_0^\alpha(E)$ as the α-packing premeasure of E. Similarly, one uses the notation $P^\alpha(E)$ for the packing α-measure of E, and $H^\alpha(E)$ for the Hausdorff α-measure of E.

Definition I.6 (Hausdorff and packing dimensions of a set, [14])**.** Let $E \subset X$. One defines the Hausdorff (packing) dimension of E as the critical point

$$\begin{aligned} \dim_H(E) &= \inf\{\alpha > 0 \mid H^\alpha(E) = 0\} \\ \dim_P(E) &= \inf\{\alpha > 0 \mid P^\alpha(E) = 0\}. \end{aligned}$$

We note that $\dim_H(X)$ or $\dim_P(X)$ may be infinite for some metric space X. One can show that, for each $E \subset X$, $\dim_H(E) \leq \dim_P(E)$ (see Theorem 3.11(h) in [14]), and this inequality is in general strict.

Definition I.7 (lower and upper packing and Hausdorff dimensions of a measure,[34])**.** Let μ be a positive Borel measure on $(X, \mathcal{B})$. The lower and upper packing and Hausdorff dimension of μ are defined, respectively, by

$$\begin{aligned} \dim_K^-(\mu) &= \inf\{\dim_K(E) \mid \mu(E) > 0,\ E \in \mathcal{B}\}, \\ \dim_K^+(\mu) &= \inf\{\dim_K(E) \mid \mu(X \setminus E) = 0,\ E \in \mathcal{B}\}, \end{aligned}$$

where K stands for H (Hausdorff) or P (packing).

Example I.1 (Skew tent map, see [57])**.** Consider the unit interval $I = [0,1]$, and for each $\lambda \in (0,1)$, define $f_\lambda : I \to I$ by the law

$$f_\lambda(x) := \begin{cases} x/\lambda & , if \quad x \in [0,\lambda], \\ (1-x)/(1-\lambda) & , if \quad x \in (\lambda, 1]. \end{cases}$$

Let $\zeta_\lambda = \{I_0, I_1\} = \{[0,\lambda], (\lambda,1]\}$ be the natural partition associated with the map f_λ. For any $(j_0,\ldots,j_n) \in \{0,1\}^n$, define a cylinder $I_{j_0,\ldots,j_n}$ by

$$I_{j_0,\ldots,j_n} = I_{j_0} \cap f_\lambda^{-1} I_{j_1} \cap \cdots \cap f_\lambda^{-n} I_{j_n}.$$

Now, for each $p \in (0,1)$, there exists a unique f_λ-invariant measure μ_p such that

$$\mu_p(I_{j_0,\ldots,j_n}) = p^m (1-p)^{n+1-m},$$

where $m = \text{card}\{k \mid j_k = 0\}$ is the number of zeros in the symbolic representation of $I_{j_0,\ldots,j_n}$. In fact, μ_p is the projection of the Bernoulli (or the product) measure with probabilities p and $1-p$, defined on the set of all infinite sequences of 0's and 1's, onto the unit interval. Note that the topological support of μ_p is the whole interval $[0,1]$. The Hausdorff dimension of the measure μ_p is given by the following formula (see [17])

$$\dim_H(\mu_p) = \frac{p\log p + (1-p)\log(1-p)}{p\log\lambda + (1-p)\log(1-\lambda)}.$$

One has $\dim_H(\mu_p) = 1$ if and only if $p = \lambda$ (in this case, the measure μ_λ is the Lebesgue measure on I).

Definition I.8. Let μ be a positive finite Borel measure over an arbitrary metric space X. One defines the upper and lower local dimensions of μ at point $x \in X$ by

$$\overline{d}_\mu(x) = \limsup_{\varepsilon\to 0} \frac{\log\mu(B(x,\varepsilon))}{\log\varepsilon} \quad \text{and} \quad \underline{d}_\mu(x) = \liminf_{\varepsilon\to 0} \frac{\log\mu(B(x,\varepsilon))}{\log\varepsilon},$$

if, for each $\varepsilon > 0$, $\mu(B(x;\varepsilon)) > 0$; if not, $\overline{\underline{d}}_\mu(x) := +\infty$. If the limit exists, one says that the measure has local dimension $d_\mu(x)$ at the point x. One says that the measure μ is exact dimensional if $d_\mu(x) = \overline{d}_\mu(x) = \underline{d}_\mu(x) = \dim_H^{\pm}(\mu)$, μ-a.e.

Definition I.9 ([17])**.** Let $F \subset \mathbb{R}^n$. Define $N(F, \delta)$ to be the least number of balls of radius δ required to cover F. The interpretation of this measure is an indication of how irregular or spread out the set is when examined at scale δ. The *upper* and *lower box-counting dimensions* are defined as

$$\begin{aligned} \underline{\dim}_B F &= \underline{\lim}_{\delta\to 0} \frac{\log N(F,\delta)}{-\log\delta} \\ \overline{\dim}_B F &= \overline{\lim}_{\delta\to 0} \frac{\log N(F,\delta)}{-\log\delta} \end{aligned}$$

Remark I.1.

i) In Euclidean spaces (like $\mathbb{R}^n$), the packing and the box counting dimensions of sets are related (see [17, 42]); namely, one has $\dim_P(F) \le \overline{\dim}_B(F)$. One can also relate the packing and the upper box counting dimension of a Borel probability measure μ over $\mathbb{R}^n$:

$$\dim_P(\mu) = \lim_{\delta\to 0} \inf\{\dim_P(F) \mid \mu(F) \ge 1-\delta\} \le \lim_{\delta\to 0} \inf\{\overline{\dim}_B(F) \mid \mu(F) \ge 1-\delta\} =: \overline{\dim}_B(\mu).$$

ii) One can also relate the Hausdorff and the upper and lower box counting dimensions of a probability measure μ in $\mathbb{R}^n$ (see [42]):

$$\dim_H(\mu) \le \underline{\dim}_B\,(\mu) \le \overline{\dim}_B\,(\mu).$$

iii) Let X be a complete separable metric space of finite multiplicity (finite topological dimension) and let μ be a Borel finite measure on X. If $\underline{d}_\mu(x) = \overline{d}_\mu(x) = d$ for μ-a.e. $x \in X$, then $\dim_H(\mu) = \dim_P(\mu) = \underline{\dim}_B(\mu) = \overline{\dim}_B(\mu) = d$ (see [42]).

Example I.2 (see Proposition 1.1, Proposition 1.3 and Theorem 1.5 in [47])**.**

1) Let $f : S^1 \to S^1$ ($S^1 = \mathbb{R}/\mathbb{Z}$) be a circle homeomorphism given by $f_\alpha(x) = (x+\alpha) \bmod 1$, with $\alpha \in \mathbb{Q}$, and let μ be an f-ergodic invariant measure. Then, $d_\mu(x) = 0$ for μ-a.e $x \in S^1$.

2) Circle diffeomorphisms with an irrational rotation number are uniquely ergodic. In this case, the properties of the invariant measure depend on how well the irrational rotation number can be approximated by rational numbers. The numbers that cannot be rapidly approximated by rationals are called Diophantine. Namely, a number τ is called Diophantine if there exist $\delta > 0$ and $K > 0$ such that $|\tau - p/q| > K/|q|^{2+\delta}$ for any integers p and q.

Let f be a C^∞ circle diffeomorphism with a Diophantine rotation number, and let μ be its unique invariant measure. Then,

$$\begin{array}{ll} \text{(i)} & d_\mu(x) = 1 \text{ for every } x \text{ in } S^1; \\ \text{(ii)} & \dim_H^-(\mu) = \dim_P^-(\mu) = 1. \end{array}$$

3) A more interesting situation occurs when the rotation number is Liouville; an irrational number τ is called a Liouville number if for any $n \geq 1$ there exist integers p and $q > 1$, such that $|\tau - p/q| < 1/q^n$.

Let τ be a Liouville number and let $0 \leq \beta \leq 1$. There exists a C^∞ circle diffeomorphism f with rotation number τ such that

$$\begin{array}{l} \text{(i) } \underline{d}_\mu(x) = \beta \text{ and } \overline{d}_\mu(x) = 1 \text{ for } \mu\text{-almost every } x \text{ in } S^1 \text{ ;} \\ \text{(ii) } \dim_H \mu = \beta \text{ and } \dim_P \mu = 1; \end{array}$$

here, μ stands for the unique f-invariant measure.

The next result presents a characterization of the lower (upper) Hausdorff and the lower (upper) packing dimensions of a probability Borel measure μ defined on a general metric space X in terms of the essential infimum (supremum) of its lower and upper local dimensions, respectively.

Proposition I.1. *Let μ be a probability measure on a metric space X. Then,*

$$\mu\text{-}\operatorname{ess\,inf} \underline{d}_\mu(x) = \dim_H^-(\mu) \leq \mu\text{-}\operatorname{ess\,sup} \underline{d}_\mu(x) = \dim_H^+(\mu),$$
$$\mu\text{-}\operatorname{ess\,inf} \overline{d}_\mu(x) = \dim_P^-(\mu) \leq \mu\text{-}\operatorname{ess\,sup} \overline{d}_\mu(x) = \dim_P^+(\mu).$$

Proof. See Appendix A. □

Poincaré's Recurrence (rates of recurrence: a quantitative description)

In what follows, (X, T) is a dynamical system such that (X, d) is a separable metric space, $\mathcal{B}(X)$ is the Borel σ-algebra of X and $T : X \to X$ is a measurable map. Let us recall some basic definitions.

Definition I.10. Let μ be a probability measure defined on the measurable space $(X, \mathcal{B}(X))$. One says that μ is T-invariant if $\mu(T^{-1}(B)) = \mu(B)$ for each $B \in \mathcal{B}(X)$. One denotes the set of T-invariant measures by $\mathcal{M}(T)$.

Definition I.11. Let μ be a probability measure defined on the measurable space $(X, \mathcal{B}(X))$. The system (T, μ) is called ergodic if $\mu \in \mathcal{M}(T)$ and if $T^{-1}(B) = B$, then $\mu(B) = 0$ or $\mu(B) = 1$. One denotes the set of ergodic measures by $\mathcal{M}_e(T)$.

One of the main problems studied in this work refers to a quantitative description of the recurrence phenomenon for a dynamical system. The recurrence problem was initially studied by Poincaré, who "stated that any dynamical system preserving a finite invariant measure exhibits a non-trivial recurrence to each set of positive measure", as mentioned by Barreira and Saussol in [6] (an information of qualitative nature).

Theorem I.1 (Poincaré's Recurrence Theorem)**.** *Let $\mu \in \mathcal{M}(T)$, and let $A \in \mathcal{B}(X)$ be such that $\mu(A) > 0$. Then, for μ-a.e. $x \in A$,* $\operatorname{card}\{n > 0 \mid T^n x \in A\} = \infty$.

This result tells us that the orbit of almost every point of A returns to A at least once (actually, it returns to A infinitely many times). It, nevertheless, does not give us any information about the first return time to A. On the other hand, Birkhoff's Ergodic Theorem states that, for almost every $x \in A$, the frequency at which the orbit of x visits A is equal to $\mu(A)$.

Definition I.12. Let (X,T) be a dynamical system, let $A \in \mathcal{B}(X)$ and let $x \in A$. One defines the first return time of x to A by

$$\tau_A(x) := \inf\{n > 0 : T^n x \in A\}.$$

A more accurate result than Poincaré's Recurrence Theorem is Kač's Theorem (more commonly known as Kač's Lemma), which proves that in the case where the measure is ergodic, the mean of the return time to A is equal to the inverse of the measure of A.

Theorem I.2 (Kač's Theorem [27])**.** *Let $\mu \in \mathcal{M}_e(T)$, and let $A \in \mathcal{B}(X)$ be such that $\mu(A) > 0$. Then,*

$$\frac{1}{\mu(A)} \int_A \tau_A(x) d\mu(x) = \frac{1}{\mu(A)}.$$

We are particularly interested in the determination of the returning rates of a given point to an arbitrarily small neighborhood of itself. Boshernitzan [9] proved a quantitative result linking the first return time to balls of small radii to the Hausdorff measure of the respective invariant measure. We note that Boshernitzan's Theorem can be reformulated in terms of the return time to balls (see [6]).

Theorem I.3 (Boshernitzan [9])**.** *Let (X,T) be a dynamical system, and assume that for $\alpha > 0$, the Hausdorff α-measure H^α is σ-finite on X. Then, for H^α-a.e. $x \in X$, one has*

$$\liminf_{n\to\infty}\{n^\beta d(x, T^n x)\} < \infty, \quad \text{with} \quad \beta = \frac{1}{\alpha}.$$

If, furthermore, $H^\alpha(X) = 0$, then for H^α-a.e. $x \in X$,

$$\liminf_{n\to\infty}\{n^\beta d(x, T^n x)\} = 0.$$

Following the idea of Boshernitzan, Barreira and Saussol [6] studied the typical (with respect to an invariant measure) asymptotic behavior of the polynomial returning rates to a ball which radius tends to zero, and showed that they were related to the local dimensions of this invariant measure (see also [2, 7, 25] for further motivations).

Definition I.13. Let $x \in X$ and let $r > 0$. One defines the first return time of x to $B(x,r)$ by the law

$$\tau_r(x) = \inf\{k \in \mathbb{N} \mid T^k x \in B(x,r)\},$$

and the lower and upper returning rates at x by

$$\underline{R}(x) = \liminf_{r \to 0} \frac{\log \tau_r(x)}{-\log r} \quad \text{and} \quad \overline{R}(x) = \limsup_{r \to 0} \frac{\log \tau_r(x)}{-\log r}.$$

Theorem I.4 (Barreira-Saussol [6]). *If $T : X \to X$ is a Borel measurable transformation, where X is a separable metric space, and if $\mu \in \mathcal{M}(T)$, then $\underline{R}(x) \leq \dim_H^+(\mu)$ for μ-a.e. $x \in X$.*

Theorem I.5 (Barreira-Saussol [6]). *If $T : X \to X$ is a Borel measurable transformation on a measurable set $X \subset \mathbb{R}^d$, $d \in N$, and if $\mu \in \mathcal{M}(T)$, then*

$$\underline{R}(x) \leq \underline{d}_\mu(x) \quad \textit{and} \quad \overline{R}(x) \leq \overline{d}_\mu(x),$$

for μ-a.e. $x \in X$.

Later, Saussol has shown in [48] that, under the hypotheses that T is a Lipschitz transformation, $h_\mu(T) > 0$ (see Appendix B for the definition of the metric entropy of a system) and that the decay of the correlations of (X, T, μ) is super-polynomial, $\underline{R}(x) = \underline{d}_\mu(x)$, and $\overline{R}(x) = \overline{d}_\mu(x)$ for μ-a.e. $x \in X$.

In this work, we present some results, for M-valued discrete stationary stochastic processes (see Definition I.18), relating $\overline{R}$ and $\underline{d}_\mu$ (namely, Theorem 1.1 and Corollary 1.1).

Another dynamical aspect of M-valued discrete stationary stochastic processes that is explored in this work refers to the quantitative waiting time indicators, defined by Galatolo in [20] as follows:

Definition I.14. Let $x, y \in X$ and let $r > 0$. The first entrance time of $\mathcal{O}(x) := \{T^i x \mid$

$i \in \mathbb{N}\}$, the T-orbit of x, into the closed ball $\overline{B}(y,r)$ is given by

$$\tau_r(x,y) = \min\{n \in \mathbb{N} \mid T^n(x) \in \overline{B}(y,r)\}.$$

Naturally, $\tau_r(x,x)$ is just the first return time into the closed ball $\overline{B}(x,r)$. The so-called quantitative waiting time indicators are defined as

$$\underline{R}(x,y) = \liminf_{r\to 0} \frac{\log \tau_r(x,y)}{-\log r} \quad \text{and} \quad \overline{R}(x,y) = \limsup_{r\to 0} \frac{\log \tau_r(x,y)}{-\log r}.$$

Suppose that $\mu \in \mathcal{M}(T) \neq \emptyset$. Then, Theorem 4 in [20] states that, for each fixed $y \in X$, one has

$$\underline{R}(x,y) \geq \underline{d}_\mu(y) \qquad \text{and} \qquad \overline{R}(x,y) \geq \overline{d}_\mu(y) \qquad \text{for } \mu\text{-a.e. } x \in X. \tag{3}$$

Furthermore, even if μ is only a probability measure on X, Theorem 10 in [20] states that for each $x \in X$, one has $\underline{R}(x,y) \geq \underline{d}_\mu(y)$ and $\overline{R}(x,y) \geq \overline{d}_\mu(y)$ for μ-a.e. $y \in X$.

Generalized fractal and correlation dimensions

As was mentioned, the Hausdorff dimension gives an information that may not be sufficient to capture the dynamical fine behaviour of the system. Thus, in order to obtain relevant information about dynamics, one should consider not only the geometry of the set Z, but also the distribution of points on Z under T. That is, one should be interested in how often a given point $x \in Z$ visits a fixed subset $Y \subset Z$ under T. If μ is an ergodic measure for which $\mu(Y) > 0$, then for a typical point $x \in Z$, the average number of visits is equal to $\mu(Y)$. Thus, the orbit distribution is completely determined by the measure μ. On the other hand, the measure μ is completely specified by the distribution of a typical orbit.

This fact is widely used in the numerical study of dynamical systems where the distributions are, in general, non-uniform and have a clearly visible fine-scaled interwoven structure of *hot* and *cold* spots, that is, regions where the frequency of visitations is either much greater than average or much less than average respectively. The distribution of hot and cold spots varies with the scale: if a small piece of the invariant set is magnified another picture of hot and cold spots can be seen.

In this direction, the so-called correlation dimension of a probability measure was introduced by Grassberger, Procaccia and Hentschel [22] in an attempt to produce a characteristic of a dynamical system that captures information about the global behavior of typical (with respect to an invariant measure) trajectories by observing only one them.

This dimension plays an important role in the numerical investigation of different dynamical systems, including those which present strange attractors. The formal definition is as follows (see [40, 41, 42]):

Definition I.15. Let (X, r) be a complete and separable (Polish) metric space, and let $T : X \to X$ be a continuous mapping. Given $x \in X$, $\varepsilon > 0$ and $n \in \mathbb{N}$, one defines the correlation sum of order $q \in \mathbb{N} \setminus \{1\}$ (specified by the points $\{T^i(x)\}$, $i = 1, \ldots, n$) by

$$C_q(x, n, \varepsilon) = \frac{1}{n^q} \operatorname{card}\{(i_1 \cdots i_q) \in \{0, 1, \cdots, n\}^q \mid r(T^{i_j}(x), T^{i_l}(x)) \leq \varepsilon \text{ for any } 0 \leq j, l \leq q\},$$

where $\operatorname{card} A$ is the cardinality of the set A. Given $x \in X$, one defines (when the limit $n \to \infty$ exists) the quantities

$$\underline{\alpha}_q(x) = \frac{1}{q-1} \varliminf_{\varepsilon \to 0} \lim_{n \to \infty} \frac{\log C_q(x, n, \varepsilon)}{\log(\varepsilon)}, \qquad \overline{\alpha}_q(x) = \frac{1}{q-1} \varlimsup_{\varepsilon \to 0} \lim_{n \to \infty} \frac{\log C_q(x, n, \varepsilon)}{\log(\varepsilon)}, \tag{4}$$

the so-called *lower* and *upper correlation dimensions of order* q *at the point* x or the *lower* and the *upper* q*-correlation dimensions at* x. If the limit $\varepsilon \to 0$ exists, we denote it by α_q, the so-called q*-correlation dimension at* x. In this case, if n is large and ε is small, one has the asymptotic relation

$$C_q(x, n, \varepsilon) \sim \varepsilon^{\alpha_q}.$$

$C_q(x, n, \varepsilon)$ gives an account of how the orbit of x, truncated at time n, "folds" into an ε-neighborhood of itself; the larger $C_q(x, n, \varepsilon)$, the more "tight" this truncated orbit is. $\underline{\alpha}_q(x)$ and $\overline{\alpha}_q(x)$ are, respectively, the lower and upper growing rates of $C_q(x, n, \varepsilon)$ as

$n \to \infty$ and $\varepsilon \to 0$ (in this order).

Let X be a general metric space and let μ be a Borel probability measure on X. For $q \in \mathbb{R}\setminus\{1\}$ and $\varepsilon \in (0,1)$, one defines the so-called *energy function* $I_{\cdot}(q,\varepsilon) : \mathcal{M} \to (0,+\infty]$ by the law

$$I_\mu(q,\varepsilon) = \int_{\text{supp}(\mu)} \mu(B(x,\varepsilon))^{q-1} d\mu(x), \tag{5}$$

where $\text{supp}(\mu)$ is the topological support of μ.

The next result shows that the two previous definitions are intimately related.

Theorem I.6 (Pesin [41, 42]). *Let X be a Polish metric space, assume that μ is ergodic and let $q \in \mathbb{N} \setminus \{1\}$. Then, there exists a set $Z \subset X$ of full μ-measure such that, for each $R, \eta > 0$ and each $x \in Z$, there exists an $N = N(x,\eta,R) \in \mathbb{N}$ such that*

$$|C_q(x,n,\varepsilon) - I_\mu(q,\varepsilon)| \le \eta$$

holds for each $n \ge N$ and each $0 < \varepsilon \le R$. In other words, $C_q(x,n,\varepsilon)$ tends to $I_\mu(q,\varepsilon)$ when $n \to \infty$ for μ-almost every $x \in X$, uniformly over $\varepsilon \in (0,R]$.

Definition I.16 (Generalized fractal dimensions). Let X be a general metric space, let μ be a Borel probability measure on X, and let $q \in (0,\infty) \setminus \{1\}$. The so-called upper and lower q-generalized fractal dimensions of μ are defined, respectively, as

$$D_\mu^+(q) = \limsup_{\varepsilon \downarrow 0} \frac{\log I_\mu(q,\varepsilon)}{(q-1)\log \varepsilon} \quad \text{and} \quad D_\mu^-(q) = \liminf_{\varepsilon \downarrow 0} \frac{\log I_\mu(q,\varepsilon)}{(q-1)\log \varepsilon}.$$

For $q = 1$, one defines the so-called upper and lower entropy dimensions (see [4] for a discussion about the connection between entropy dimensions and Rényi information dimensions), respectively, as

$$D_\mu^+(1) = \limsup_{\varepsilon \downarrow 0} \frac{\int_{\text{supp}(\mu)} \log \mu(B(x,\varepsilon)) d\mu(x)}{\log \varepsilon},$$

$$D_\mu^-(1) = \liminf_{\varepsilon \downarrow 0} \frac{\int_{\text{supp}(\mu)} \log \mu(B(x,\varepsilon)) d\mu(x)}{\log \varepsilon}.$$

Example I.3 (see [57] and Example I.1)**.** Consider again the skew tent map, with $\lambda = 1/2$ and μ_p, for $p \in (0,1)$, as in Example I.1. Assume also that $p > 1/2$. Using the natural partition $\zeta = \{I_0, I_1\} = \{[0,1/2],(1/2,1]\}$, one can show by a direct computation that the generalized upper and lower dimensions are

$$D_\mu^{\pm}(q) = -\frac{1}{q-1}\frac{\log(p^q + (1-p)^q)}{\log 2},\ q \neq 1, \quad \text{and} \quad D_\mu^{\pm}(1) = -\frac{p\log p + (1-p)\log(1-p)}{\log 2}.$$

Some useful relations involving the generalized fractal, Hausdorff and packing dimensions of a probability measure are given by the following inequalities, which combine Propositions 4.1 and 4.2 in [4] with Proposition I.1 (although Propositions 4.1 and 4.2 in [4] were originally proved for probability measures defined on $\mathbb{R}$, one can extend them to probability measures defined on a general metric space X; see also [46]).

Proposition I.2 ([4, 46])**.** *Let μ be a Borel probability measure over X, let $q > 1$ and let $0 < s < 1$. Then,*

$$D_\mu^-(q) \leq \dim_H^-(\mu) \leq \dim_H^+(\mu) \leq D_\mu^-(s),$$
$$D_\mu^+(q) \leq \dim_P^-(\mu) \leq \dim_P^+(\mu) \leq D_\mu^+(s).$$

Moreover, $D_\mu^{\pm}(q) \leq D_\mu^{\pm}(1) \leq D_\mu^{\pm}(s)$.

Contributions and organization of the present thesis

In this work, we are interested in the characterization of some typical (in Baire's sense) dimensional properties of invariant measures, where the set of invariant measures is endowed with the weak topology. We start with shift-type systems, for which, as will be seen, the chaotic dynamics is associated with some rather exotic invariant measures. Later, we also study dimensional properties of invariant measures for other dynamical systems, such as Axiom A systems.

Recall that a subset $\mathcal{R}$ of a topological space X is residual if it contains the intersection of a countable family, $\{U_k\}$, of open and dense subsets of X. A topological space X is a Baire space if every residual subset of X is dense in X. By the Baire Category Theorem, every complete metric space is a Baire space.

Definition I.17. A property $\mathbb{P}$ is said to be generic in the space X if there exists a residual subset $\mathcal{R}$ of X such that each $x \in \mathcal{R}$ satisfies property $\mathbb{P}$.

Note that, given a countable family of generic properties $\mathbb{P}_1, \mathbb{P}_2, \cdots$, all of them are simultaneously generic in X. This is because the family of residual sets is closed under countable intersections.

Hausdorff and packing dimension - rates of recurrence

Let (M, ρ) be a complete separable (Polish) metric space, and let S be its σ-algebra of Borel sets. Now, define $(X, \mathcal{B})$ as the bilateral product of a countable number of copies of (M, S). Note that $\mathcal{B}$ coincides with the σ-algebra of the Borel sets in the product topology. Let d be any metric in $X = M^{\mathbb{Z}}$ which is compatible with the product topology (that is, d induces an equivalent topology). It is straightforward to show that (X, d) is also a Polish metric space.

One can define in X the so-called full-shift operator, T, by the action

$$Tx = y,$$

where $x = (\ldots, x_{-n}, \ldots, x_n, \ldots)$, $y = (\ldots, y_{-n}, \ldots, y_n, \ldots)$, and for each $i \in \mathbb{Z}$, $y_i = x_{i-1}$. T is clearly a homeomorphism of X into itself. We choose d in such a way that T and T^{-1} are Lipschitz transformations; set, for instance, for each $x, y \in X$,

$$d(x,y) = \sum_{|n| \geq 0} \frac{1}{2^{|n|}} \frac{\rho(x_n, y_n)}{1 + \rho(x_n, y_n)}. \tag{6}$$

Let $\mathcal{M}(T)$ be the space of all T-invariant probability measures, endowed with the weak topology (that is the coarsest topology for which the net $\{\mu_\alpha\}$ converges to μ if, and only if, for each bounded and continuous function f, $\int f d\mu_\alpha \to \int f d\mu$). Since X is Polish, $\mathcal{M}(T)$ is also a Polish metrizable space (see [15]).

Definition I.18. Given $\mu \in \mathcal{M}(T)$, the triple (X, T, μ) is called an M-valued discrete stationary stochastic process (see [39, 51, 53]; see also [19] for a discussion of the role of such systems in the study of continuous self-maps over general metric spaces).

In **Chapter I** we obtain several results, compiled in Theorem 1.1 (items I-VIII), where we characterize typical (generic) dimensional properties of invariant measures associated with an M-valued discrete stationary stochastic process, for M a perfect and separable metric space. More specifically, we show that the set of invariant measures with upper Hausdorff dimension equal to zero and lower packing dimension equal to infinity is a dense G_δ subset of $\mathcal{M}(T)$. We also show that the set of invariant measures with upper rate of recurrence equal to infinity and lower rate of recurrence equal to zero is a G_δ subset of $\mathcal{M}(T)$. Furthermore, we show that the set of invariant measures with upper quantitative waiting time indicator equal to infinity and lower quantitative waiting time indicator equal to zero is residual in $\mathcal{M}(T)$.

Generalized fractal and correlation dimension - rates of recurrence

In **Chapter II** we show that, for topological dynamical systems with a dense set (in the weak topology) of periodic measures, a typical (in Baire's sense) invariant measure has, for each $q > 0$, zero lower q-generalized fractal dimension. This implies, in particular, that a typical invariant measure has zero upper Hausdorff dimension and zero lower rate of recurrence. Of special interest is the M-valued discrete stationary stochastic process (for the case where $X = M^{\mathbb{Z}}$ is endowed with a sub-exponential metric and the alphabet M is a perfect and compact metric space), for which we show that a typical invariant measure has, for each $q > 1$, infinite upper q-correlation dimension. Under the same conditions, we show that a typical invariant measure has, for each $s \in (0, 1)$ and each $q > 1$, zero lower

s-generalized and infinite upper q-generalized dimensions.

More specifically, we show in Section 2.1 that for each $s \in (0,1)$ and each $q > 1$, both $D := \{\mu \in \mathcal{M} \mid d_\mu^-(s) = 0\}$ (see Proposition 2.1 for a definition of $d_\mu^-(s)$) and $CD := \{\mu \in \mathcal{M} \mid D_\mu^+(q) = +\infty\}$ are G_δ sets. In Section 2.2, we show that these sets are dense in $\mathcal{M}(T)$.

Axiom A systems and expansive homeomorphisms

We are also interested in dimensional properties of invariant measures for Axiom A systems and expansive homeomorphisms.

Suppose that M is a compact C^∞ Riemannian manifold, and that $f : M \to M$ is a diffeomorphism. Then, the derivative of f can be seen as a map $df : TM \to TM$, where $TM = \bigcup_{x \in M} T_x M$ is the tangent bundle of M and $df_x : T_x M \to T_{f(x)} M$.

Definition I.19. A closed subset $\Lambda \subset M$ is said to be hyperbolic if $f(\Lambda) = \Lambda$, and if each tangent space $T_x M$, $x \in \Lambda$, can be written as a direct sum $T_x M = E_x^u \oplus E_x^s$ of subspaces so that

(a) $df_x(E_x^s) = E_{f(x)}^s$, $df_x(E_x^u) = E_{f(x)}^u$;

(b) there exist constants $c > 0$ and $\lambda \in (0,1)$ so that

$$\|df_x^n(v)\| \le c\lambda^n \|v\| \quad \text{if } v \in E_x^s,\ n \ge 0,$$

and

$$\left\|df_x^{-n}(v)\right\| \le c\lambda^n \|v\| \quad \text{if } v \in E_x^u,\ n \ge 0;$$

(c) E_x^s, E_x^u vary continuously with x.

A point $x \in M$ is said to be *non-wandering* if, for each neighborhood U of x,

$$U \cap \bigcup_{n > 0} f^n U \ne \emptyset.$$

The set $\Omega = \Omega(f)$ of all non-wandering points is closed and f-invariant. A point x is periodic if $f^n x = x$ for some $n > 0$; clearly, each periodic point belongs to Ω.

Definition I.20. f is called an Axiom A diffeomorphism if Ω is a hyperbolic set and the set of periodic points is dense in Ω.

Remark I.2. Smale's Spectral Decomposition Theorem (see [55]) states that if f satisfies the Axiom A conditions, then one can write $\Omega = \Omega_1 \cup \cdots \Omega_M$, where the Ω_l are disjoint closed f-invariant sets and $f \restriction_{\Omega_l}$ is topologically transitive. In what follows, let $T : X \to X$, where $X := \Omega_l$ and $T := f \restriction_{\Omega_l}$; the dynamical system (T, X) will be called an Axiom A system.

Let us recall the definition of an expansive map.

Definition I.21. Let X be a metrizable space and let $f : X \to X$ be a homeomorphism. f is said to be expansive if there exists a $\delta > 0$ such that, for each pair of different points $x, y \in X$, there exists an $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) > \delta$, where d is any metric which induces the topology of X.

Note that expansivity, defined in Definition I.21, is a topological notion, i.e., it does not depend on the choice of a particular metric under consideration (compatible with the topology, of course), although the expansivity constant δ may depend on d.

Remark I.3. Equivalently, f is expansive if there exists a $\delta > 0$ such that, for each $x \in X$, $\Gamma(\delta(x)) = \{x\}$, where

$$\Gamma(\delta(x)) = \{y \in X : d(f^i(x), f^i(y)) \leq \delta, \forall i \in \mathbb{Z}\}.$$

At last, we recall the definition of an f-homogeneous measure and present some examples of such measures.

Definition I.22. Let f be a continuous transformation of a compact metric space (X, d). A Borel probability measure μ on X is said to be f-homogeneous if for each $\varepsilon > 0$, there exist $\delta > 0$ and $c > 0$ such that, for each $n \in \mathbb{N}$ and each $x, y \in X$,

$$\mu(B(y, n, \delta)) \leq c\, \mu(B(x, n, \varepsilon)), \tag{7}$$

where $B(x, n, \varepsilon) := \{y \in X \mid d(f^i(x), f^j(y)) < \varepsilon, \forall i = 0, \ldots, n\}$ is the Bowen ball of size n and radius ε, centered at x.

In **Chapter III** we present, under some hypotheses over the local dimensions of Borel probability measures over compact metric spaces, upper and lower bound to the generalized fractal dimensions of such measures. Based on this result, we show a version of Young's Theorem for the generalized fractal dimensions of homogeneous measures of $C^{1+\alpha}$-Axiom A systems ($\alpha > 0$). Furthermore, for Axiom A systems, we show that the set of invariant measures with zero correlation dimension is a generic set in the set of all invariant measures, under a hyperbolic metric.

Finally, in **Appendix A** we give the proof of Proposition I.1. In **Appendix B-E** we present some basic results and definitions which are necessary for the good understanding of this thesis.

CHAPTER I

GENERIC PROPERTIES OF INVARIANT MEASURES OF FULL-SHIFT SYSTEMS OVER PERFECT SEPARABLE METRIC SPACES

Some generic properties (in Baire's sense; see Definition I.17) of the invariant measures of the M-valued discrete stationary stochastic processes presented in Definition I.18, like ergodicity and zero entropy (this one in case $M = \mathbb{R}$), have been studied by Parthasarathy in [39] and Sigmund in [51, 53], respectively. In the last decade, various studies about the full-shift system over an uncountable alphabet have been performed; more specifically, we can mention the works about the Gibbs state in Ergodic Optimization [3], entropy and the variational principle for one-dimensional lattice systems [32]. All results in this Chapter appear in our article published in Stochastic and Dynamics [12], in January 2021.

We begin this chapter making some comments and observations about the results obtained here, including their dynamical and topological consequences. The proof of Theorem 1.1 is presented in Sections 1.1 and 1.2.

Theorem 1.1. *Let $(X, T, \mathcal{B})$ be the full-shift dynamical system over $X = \prod_{-\infty}^{+\infty} M$, where the alphabet M is a perfect and separable metric space. Then:*

I. *The set of ergodic measures, $\mathcal{M}_e$, is residual in $\mathcal{M}(T)$.*

II. *The set of invariant measures with full support, C_X, is a dense G_δ subset of $\mathcal{M}(T)$.*

III. *The set $HD = \{\mu \in \mathcal{M}(T) \mid \dim_H^+(\mu) = 0\}$ is a dense G_δ subset of $\mathcal{M}(T)$.*

IV. *The set $PD = \{\mu \in \mathcal{M}(T) \mid \dim_P^-(\mu) = +\infty\}$ is a dense G_δ subset of $\mathcal{M}(T)$.*

V. *The set* $\underline{\mathcal{R}} = \{\mu \in \mathcal{M}(T) \mid \underline{R}(x) = 0$, *for* μ*-a.e.* $x\}$ *is a dense* G_δ *subset of* $\mathcal{M}(T)$.

VI. *The set* $\overline{\mathcal{R}} = \{\mu \in \mathcal{M}(T) \mid \overline{R}(x) = +\infty$, *for* μ*-a.e.* $x\}$ *is a dense* G_δ *subset of* $\mathcal{M}(T)$.

VII. *The set* $\underline{\mathscr{R}} = \{\mu \in \mathcal{M}(T) \mid \underline{R}(x,y) = 0$, *for* $(\mu \times \mu)$*-a.e.* $(x,y) \in X \times X\}$ *is a dense* G_δ *subset of* $\mathcal{M}(T)$.

VIII. *The set* $\overline{\mathscr{R}} = \{\mu \in \mathcal{M}(T) \mid \overline{R}(x,y) = +\infty$, *for* $(\mu \times \mu)$*-a.e.* $(x,y) \in X \times X\}$ *is residual in* $\mathcal{M}(T)$.

Item I was proved by Oxtoby in [38] and Parthasarathy in [39], using the fact that the set of T-periodic or T-closed orbit measures (that is, measures of the form $\frac{1}{k_x}\sum_{i=0}^{k_x-1}\delta_{T^i x}(\cdot)$, where x is a T-periodic point of period k_x) is dense in $\mathcal{M}(T)$. Sigmund has proved item II in [53] (see also [15]). We have opted to include these results in Theorem 1.1 (the proofs of some of these results, among others, are presented in Appendix D) since they are used in the proofs of items III-VIII, which comprise our main contributions to the problem.

As a direct consequence of Theorem 1.1, we have obtained for typical ergodic measures, that is, for $\mu \in \mathcal{RD} = \underline{\mathcal{R}} \cap \overline{\mathcal{R}} \cap PD \cap HD$, some relations between $\overline{\underline{R}}$ and $\overline{\underline{d}}_\mu$ which are similar to those obtained by Saussol and Barreira (see [6] and [49]).

Corollary 1.1. *Let M be a compact and perfect metric space, and let $\mu \in \mathcal{RD} \subset \mathcal{M}_e(T)$. Then,* $\underline{R}(x) = \underline{d}_\mu(x) = 0$ *and* $\overline{R}(x) = \overline{d}_\mu(x) = \infty$, *for* μ*-a.e.* $x \in X$.

It follows from items III and IV in Theorem 1.1 that there exists a dense G_δ set, $\mathcal{D} := PD \cap HD \subset \mathcal{M}_e(T)$, such that each $\mu \in \mathcal{D}$ is somewhat similar, in one hand, to a "uniformly distributed" measure, whose lower packing dimension is maximal (for instance, when $X = [0,1]^{\mathbb{Z}}$, the shift Bernoulli measure $\Lambda = \prod_{-\infty}^{+\infty}\lambda$, where λ is the Lebesgue measure on $[0,1]$, is an uniformly distributed measure, whose lower packing dimension is infinite) and in the other hand, to a purely point measure, whose upper Hausdorff dimension is zero.

Moreover, by Definition I.7, each $\mu \in \mathcal{D}$ is supported on a Borel set $Z = Z(\mu)$ such that $\dim_{top}(Z) \le \dim_H(Z) = 0 < \dim_P(Z) = \infty$, where $\dim_{top}(Z)$ stands for the topological dimension of Z (see, [26] Sect. 4, page 107, for a proof of the inequality $\dim_{top}(Z) \le$

$\dim_H(Z)$; see also Definition I.2). Since $\dim_{top}(Z) = 0$, if one also assumes that $\mathrm{supp}(\mu) = X$ (just take $\mu \in C_X(T) \cap \mathcal{D}$), one concludes that Z is a dense and totally disconnected set in X with zero Hausdorff and infinite packing dimensions. Furthermore, one may take Z as a dense G_δ subset of X (see Proposition 1.7).

Remark 1.1. It is worth noting that although the packing dimension of X is infinite (since, for each $\mu \in \mathcal{D}$, $\dim_P(Z) = \infty$), its topological dimension may be finite. This is not unexpected, altogether: there are examples of topological spaces where $\dim_P(X) > \dim_{top}(X)$. Indeed, let σ be the full-shift on the space $\Sigma_p = \{0, 1, \ldots, p-1\}^{\mathbb{Z}}$ of two-sided infinite sequences of symbols $0, 1, \ldots, p-1$. We assume that Σ_p is endowed with the standard metric

$$\rho\left(\omega^{(1)}, \omega^{(2)}\right) = \sum_{i=-\infty}^{\infty} \frac{\left|\omega_i^{(1)} - \omega_i^{(2)}\right|}{a^{|i|}},$$

where $a > 1$. Then, for any σ-invariant measure μ on Σ_p and μ-a.e. $x \in \Sigma_p$,

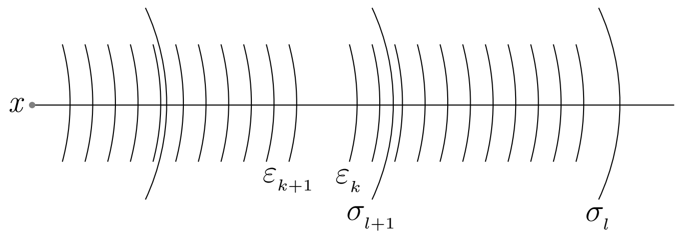
$$\underline{d}_\mu(x) = \overline{d}_\mu(x) = \frac{h_\mu(\sigma)}{\log a}.$$

Thus, one has $\dim_P(\Sigma_p) \geq \dim_P^+(\mu) = \frac{h_\mu(\sigma)}{\log a} > \dim_{top}(\Sigma_p) = 0$.

Items V and VI in Theorem 1.1 say that there exists a dense G_δ set, $\mathcal{R} := \underline{\mathcal{R}} \cap \overline{\mathcal{R}} \subset \mathcal{M}_e(T)$, of ergodic measures such that if $\mu \in \mathcal{R}$, then there exists a Borel set Z, with $\mu(Z) = 1$, so that if $x \in Z$, then $\underline{R}(x) = 0$ and $\overline{R}(x) = \infty$. This means that given a very large α and a very small β, for each $x \in Z$, one has $\underline{R}(x) \leq \beta$ and $\overline{R}(x) \geq \alpha$. So, there exist sequences (ε_k) and (σ_l) converging to zero such that, for each $k, l \in \mathbb{N}$, $\tau_{\varepsilon_k}(x) \leq \varepsilon_k^{-\beta}$ and $\tau_{\sigma_l}(x) \geq \sigma_l^{-\alpha}$, respectively. Setting, for each $k, l \in \mathbb{N}$, $s_k = 1/\varepsilon_k$ and $t_l = 1/\sigma_l$, one has $\tau_{1/s_k}(x) \leq s_k^\beta$ and $\tau_{1/t_l}(x) \geq t_l^\alpha$, respectively.

Therefore, given $x \in Z$, there exists a time sequence (time scale) for which the first incidence of $\mathcal{O}(x)$ to one of its spherical neighborhoods (which depend on time) occurs as fast as possible (that is, it is of order 1; this means that the first return time to those neighborhoods increases sub-polynomially fast); accordingly, there exists a time sequence for which the first incidence of $\mathcal{O}(x)$ to one of its spherical neighborhoods increases as fast as possible (that is, super-polynomially fast).

The following scheme tries to depict how subsequent elements of both sequences are related. Between to consecutive elements of (σ_k), there are several elements of (ε_l):



Here, we also show that the typical measures described in Theorem 1.1 are supported on the dense G_δ set $\mathfrak{R} = \{x \in X \mid \underline{R}(x) = 0 \text{ and } \overline{R}(x) = \infty\}$ (Proposition 1.11).

Finally, combining items II, VII and VIII of Theorem 1.1, one concludes that for a typical measure $\mu \in \overline{\mathscr{R}} \cap \underline{\mathscr{R}} \cap C_X(T)$, almost every T-orbit $\mathcal{O}(x)$ densely fills the whole space (given that μ is supported on a dense subset of X and $\underline{R}(x, y) = 0$ for $(\mu \times \mu)$-a.e $(x, y) \in X \times X$), but not in a homogeneous fashion. Namely, as in the previous analysis, there exists a time scale for which the first entrance time of $\mathcal{O}(x)$ to one of the spherical neighborhoods (which depend on time) of y is of order 1; accordingly, there exists a time sequence for which the first entrance time of the $\mathcal{O}(x)$ to one of the spherical neighborhoods of y increases as fast as possible. Naturally, these time scales depend on the pair $(x, y) \in X \times X$.

Remark 1.2.

i) It is true that the sets defined in items III to VIII of Theorem 1.1 are G_δ subsets of $\mathcal{M}(T)$ for any topological dynamical system (X, T)[1] such that X is Polish and both T, T^{-1} are Lipshitz transformations (this is particularly true for Axiom A systems on smooth compact Riemannian n-manifolds, (M, T), where $f : M \to M$ is a diffeomorphism: it is possible to show that both f and f^{-1} are Lipschitz with respect to the natural the Riemannian metric; see Theorem 5.1 in [18]).

ii) It is also true that the sets defined in items III, V and VII of Theorem 1.1 are dense in $\mathcal{M}(T)$ for any topological dynamical system (X, T) such that X is a separable metric

[1]By a topological dynamical system we understand a pair (X, T) such that X is a Polish metric space and $T : X \to X$ is a continuous transformation.

space and the set of T-periodic measures is dense in $\mathcal{M}(T)$. This is particularly true for any system satisfying the specification property (see [53] for a proof of this proposition and examples of systems that satisfy this property; see also [43] and Appendix B for the definition of this property), or even milder conditions (see [21, 24, 28, 30, 31] for a broader discussion involving such conditions).

iii) Since the Axiom A systems described in Introduction also have a dense set of T-periodic measures (here, X stands for a closed f-invariant set and $T := f \upharpoonright X$ is topologically transitive; see [50]), one may combine both properties and obtain the following result.

Theorem 1.2. *Let (X, T) be an Axiom A system as described in Introduction. Then, the set $\{\mu \in \mathcal{M}_e(T) \mid \underline{R}(x, y) = 0,$ for $(\mu \times \mu)$-a.e. $(x, y) \in X \times X\}$ is a dense G_δ subset of $\mathcal{M}_e$.*

iv) The hypothesis that the alphabet M is perfect is crucial for items IV and VI of Theorem 1.1. Namely, the fact that M does not have isolated points is required to guarantee that one can always choose the periodic point x of period s in the statement of Lemma 1.6 so that $x_i \neq x_j$ if $i \neq j$, $1 \leq i, j \leq s$. This result, whose proof relies on the product structure of X, is required in the proof of Proposition 2.7, which in turn guarantees that the sets presented in items IV and VI of Theorem 1.1 are dense. Indeed, our strategy depends on the fact that the set of ergodic measures with arbitrarily large entropy is dense in $\mathcal{M}_e(T)$; here, we explicitly use the fact that the lower packing dimension of an ergodic measure is lower bounded by (up to a constant) its entropy (see Lemma 1.4).

1.1 Sets of ergodic measures with zero Hausdorff and infinity packing dimensions

1.1.1 G_δ sets

Let $\mathcal{M}(X)$ (with simple notation $\mathcal{M}$) be the space of probability measures defined on the measurable space $(X, \mathcal{B}(X))$, endowed with the topology of the weak convergence. Let, for each $x \in X$ and each $\varepsilon > 0$,

$$\chi_{_{B(x,\varepsilon)}}(y) = \begin{cases} 1 & , if \quad d(x,y) < \varepsilon, \\ \\ 0 & , if \quad d(x,y) \geq \varepsilon, \end{cases}$$

and note that for each $\mu \in \mathcal{M}$, $\mu(B(x,\varepsilon)) = \int \chi_{B(x,\varepsilon)}(y) d\mu(y)$. Since, for each $x \in X$ and each $\varepsilon > 0$, $\chi_{_{B(x,\varepsilon)}} : X \to [0,1]$ is not necessarily continuous, one needs to approximate, for each $\varepsilon > 0$, the mapping $\mathcal{M} \times X \ni (\mu, x) \mapsto \mu(B(x,\varepsilon)) \in [0,1]$ (in the product topology of $\mathcal{M} \times X$) by a continuous one. This motivates the follows results.

The first one gives a continuous approximation of the characteristic function of the ball of center x and radius ε.

Lemma 1.1. *Define, for each $x \in X$ and each $\varepsilon > 0$, the application $f_x^\varepsilon : X \to [0,1]$ by the law*

$$f_x^\varepsilon(y) := \begin{cases} 1 & , if \quad d(x,y) \leq \varepsilon, \\ \\ -\dfrac{d(x,y)}{\varepsilon} + 2 & , if \quad \varepsilon \leq d(x,y) \leq 2\varepsilon, \\ \\ 0 & , if \quad d(x,y) \geq 2\varepsilon. \end{cases}$$

Given any $\eta > 0$, there exists $\delta > 0$ such that if $d(w,x) < \delta$, then $|f_w^\varepsilon(y) - f_x^\varepsilon(y)| < \eta$ for all $y \in X$. In particular, if $\{x_n\}$ is a sequence in X with limit $x \in X$, then the sequence of functions $\{f_{x_n}^\varepsilon\}$ converges uniformly to f_x^ε over X.

Proof. The proof is split in three cases:

CASE 1: Let $y \in X$ be such that $d(x, y) \leq \varepsilon$ and:

a) $d(w, y) \leq \varepsilon$. In this case, one has $|f_w^\varepsilon(y) - f_x^\varepsilon(y)| = 0$, and one can take any $\delta > 0$.

b) $\varepsilon < d(w, y) \leq 2\varepsilon$. For this case, one has $|f_w^\varepsilon(y) - f_x^\varepsilon(y)| = \frac{d(w,y)}{\varepsilon} - 1$. In order to obtain $\frac{d(w,y)}{\varepsilon} - 1 < \eta$, just take $d(w, x) < \varepsilon\eta$. Thus, $d(w, y) \leq d(w, x) + d(x, y) < \varepsilon\eta + \varepsilon$.

c) $d(w, y) \geq 2\varepsilon$. This case is impossible; let $d(x, w) < \varepsilon/2$ and note that $2\varepsilon \leq d(w, y) \leq d(w, x) + d(x, y) < \varepsilon/2 + \varepsilon$, an absurd.

Thus, in this case, just set $\delta = \varepsilon \min\{1/2, \eta\}$.

CASE 2: Let $y \in X$ be such that $\varepsilon < d(x, y) \leq 2\varepsilon$ and:

a) $d(w, y) \leq \varepsilon$. In this case, one has $|f_w^\varepsilon(y) - f_x^\varepsilon(y)| = \frac{d(x,y)}{\varepsilon} - 1$. In order to obtain $\frac{d(x,y)}{\varepsilon} - 1 < \eta$, just let $d(w, x) < \varepsilon\eta$. Thus, $d(x, y) \leq d(w, x) + d(w, y) < \varepsilon\eta + \varepsilon$.

b) $\varepsilon < d(w, y) \leq 2\varepsilon$. In this case, one has $|f_w^\varepsilon(y) - f_x^\varepsilon(y)| = \frac{1}{\varepsilon}|d(w, y) - d(x, y)|$. In order to obtain $\frac{1}{\varepsilon}|d(w, y) - d(x, y)| < \eta$, just let $d(w, x) < \varepsilon\eta$. Namely, note that $d(y, w) \leq d(y, x) + d(x, w)$ and $d(y, x) \leq d(y, w) + d(w, x)$, so $-d(w, x) \leq d(w, y) - d(x, y) \leq d(w, x)$.

c) $d(w, y) \geq 2\varepsilon$. In this case, one has $|f_w^\varepsilon(y) - f_x^\varepsilon(y)| = 2 - \frac{d(x,y)}{\varepsilon}$. In order to obtain $2 - \frac{d(x,y)}{\varepsilon} < \eta$, just let $d(w, x) < \varepsilon\eta$. Thus, $d(x, y) \geq d(w, y) - d(w, x) > 2\varepsilon - \varepsilon\eta$.

Therefore, set $\delta = \varepsilon\eta$ in this case.

CASE 3: Let $y \in X$ such that $d(x, y) \geq 2\varepsilon$ and:

a) $d(w, y) \leq \varepsilon$. This case is impossible; let $d(x, w) < \varepsilon/2$ and note that $2\varepsilon \leq d(x, y) \leq d(y, w) + d(w, x) < \varepsilon + \varepsilon/2$, an absurd.

b) $\varepsilon < d(w, y) \leq 2\varepsilon$. In this case, one has $|f_w^\varepsilon(y) - f_x^\varepsilon(y)| = 2 - \frac{d(x,y)}{\varepsilon}$. In order to obtain $2 - \frac{d(x,y)}{\varepsilon} < \eta$, just take $d(w, x) < \varepsilon\eta$. Namely, note that $d(y, x) \leq d(y, w) + d(x, w)$ implies $-d(w, y) \leq d(w, x) - d(y, x) \leq \varepsilon\eta - 2\varepsilon$.

c) $d(w, y) \geq 2\varepsilon$. In this case, one has $|f_w^\varepsilon(y) - f_x^\varepsilon(y)| = 0$, so one can take any $\delta > 0$.

Thus, set $\delta = \varepsilon\eta$ in this case.

Therefore, one can set $\delta = \varepsilon \min\{1/2, \eta\}$ in order to prove the result. The uniform convergence of the sequence of functions is immediate. □

Lemma 1.2. *Let $\mu \in \mathcal{M}$. Then, for each $x \in X$,*

$$\underline{d}_\mu(x) = \liminf_{\varepsilon \to 0} \frac{\log f_{x,\varepsilon}(\mu)}{\log \varepsilon} \qquad \text{and} \qquad \overline{d}_\mu(x) = \limsup_{\varepsilon \to 0} \frac{\log f_{x,\varepsilon}(\mu)}{\log \varepsilon},$$

where, for each $x \in X$ and each $\varepsilon > 0$,

$$f_{x,\varepsilon}(\,\cdot\,) : \mathcal{M} \to [0,1] \quad \text{is defined by the law} \quad f_{x,\varepsilon}(\mu) := \int f_x^\varepsilon(y) d\mu(y),$$

and $f_x^\varepsilon : X \to [0,1]$ is defined as in Lemma 1.1. Furthermore, the function $f_\varepsilon(\mu, x) = f_{x,\varepsilon}(\mu) : \mathcal{M} \times X \to [0,1]$ is jointly continuous.

Proof. It follows from the definition of f_x^ε that, for each $x \in X$ and each $\varepsilon > 0$, $f_{x,\varepsilon/2}(\mu) \leq \mu(B(x,\varepsilon)) \leq f_{x,2\varepsilon}(\mu)$. Then, if $\mu(B(x;\varepsilon)) > 0$, one has $\frac{\log f_{x,\varepsilon/2}(\mu)}{\log \varepsilon} \geq \frac{\log \mu(B(x,\varepsilon))}{\log \varepsilon} \geq \frac{\log f_{x,2\varepsilon}(\mu)}{\log \varepsilon}$, which proves the first assertion. If $\mu(B(x;\varepsilon)) = 0$, given that $f_{x,\varepsilon/2}(\mu) \leq \mu(B(x,\varepsilon))$, just set $\limsup_{\varepsilon \to 0}(\inf) \frac{\log f_{x,\varepsilon}(\mu)}{\log \varepsilon} = +\infty$.

Note that, for each $x \in X$ and each $\varepsilon > 0$, $f_x^\varepsilon : X \to \mathbb{R}$ is a continuous function such that, for each $y \in X$, $\chi_{B(x,\varepsilon/2)}(y) \leq f_x^\varepsilon(y) \leq \chi_{B(x,\varepsilon)}(y)$. Given that $f_x^\varepsilon(y)$ depends only on $d(x, y)$, Lemma 1.1 show that $f_{x_n}^\varepsilon$ converges uniformly to f_x^ε on X when $x_n \to x$.

We combine this remark with Theorems E.1 and E.2 (see Appendix D) in order to prove that $f_\varepsilon(\mu, x)$ is jointly continuous. Let (μ_m) and (x_n) be sequences in $\mathcal{M}$ and X, respectively, such that $\mu_m \to \mu$ and $x_n \to x$. Firstly, we show that

$$\lim_{m \to \infty} \lim_{n \to \infty} f_\varepsilon(\mu_m, x_n) = \lim_{m \to \infty} \lim_{n \to \infty} \int f_{x_n}^\varepsilon(y) d\mu_m(y) = f_\varepsilon(\mu, x).$$

Since, for each $y \in X$, $|f_{x_n}^\varepsilon(y)| \leq 1$, it follows from Dominated Convergence Theorem that, for each $m \in \mathbb{N}$, $\lim\limits_{n\to\infty} \int f^{\varepsilon}_{x_n}(y) d\mu_m(y) = \int f^{\varepsilon}_{x}(y) d\mu_m(y)$. Now, since f^{ε}_x is continuous, it follows from the definition of weak convergence that

$$\lim_{m\to\infty} \lim_{n\to\infty} \int f^{\varepsilon}_{x_n}(y) d\mu_m(y) = \lim_{m\to\infty} \int f^{\varepsilon}_{x}(y) d\mu_m(y) = f_{\varepsilon}(\mu, x).$$

The next step consists in showing that, for each $n \in \mathbb{N}$, the function $\varphi_n : \mathbb{N} \to \mathbb{N}$, defined by the law $\varphi_n(m) := f_{\varepsilon}(\mu_m, x_n)$, converges uniformly in $m \in \mathbb{N}$ to $\varphi(m) := \lim\limits_{n\to\infty} f_{\varepsilon}(\mu_m, x_n) = \int f^{\varepsilon}_x(y) d\mu_m(y)$. Let $\delta > 0$ and fix $m \in \mathbb{N}$. Since $f^{\varepsilon}_{x_n}(y)$ converges uniformly to $f^{\varepsilon}_x(y)$, there exists $N \in \mathbb{N}$ such that, for each $n \geq N$ and each $y \in X$, $\left| f^{\varepsilon}_{x_n}(y) - f^{\varepsilon}_x(y) \right| < \delta$. Then, one has, for each $n \geq N$,

$$\begin{aligned} |\varphi_n(m) - \varphi(m)| = \left| \int f^{\varepsilon}_{x_n}(y) d\mu_m(y) - \int f^{\varepsilon}_x(y) d\mu_m(y) \right| \ &\leq \ \int \left| f^{\varepsilon}_{x_n}(y) - f^{\varepsilon}_x(y) \right| d\mu_m(y) \\ &< \ \delta. \end{aligned}$$

It follows from Theorem E.2 that $\lim\limits_{n,m\to\infty} f_{\varepsilon}(\mu_m, x_n) = f_{\varepsilon}(\mu, x)$. Given that $\lim\limits_{n\to\infty} f_{\varepsilon}(\mu_m, x_n) = \int f^{\varepsilon}_x(y) d\mu_m(y)$ and that $\lim\limits_{m\to\infty} f_{\varepsilon}(\mu_m, x_n) = \int f^{\varepsilon}_{x_n}(y) d\mu(y)$ exist for each $m \in \mathbb{N}$ and each $n \in \mathbb{N}$, respectively, Theorem E.1 implies that

$$\lim_{m\to\infty} \lim_{n\to\infty} f_{\varepsilon}(\mu_m, x_n) = \lim_{n\to\infty} \lim_{m\to\infty} f_{\varepsilon}(\mu_m, x_n) = \lim_{n,m\to\infty} f_{\varepsilon}(\mu_m, x_n) = f_{\varepsilon}(\mu, x).$$

Hence, if (μ_n, x_n) is some sequence in $\mathcal{M} \times X$ (endowed with the product topology) such that $(\mu_n, x_n) \to (\mu, x)$, then $\lim\limits_{n\to\infty} f_{\varepsilon}(\mu_n, x_n) = f_{\varepsilon}(\mu, x)$, showing that $f_{\varepsilon}(\,\cdot\,,\,\cdot\,) = f_{\cdot,\varepsilon}(\,\cdot\,)$ is jointly continuous at (μ, x). □

For each $t > 0$, let $\varepsilon = 1/t$. Since, for each $x \in X$,

$$\overline{\underline{d}}_{\mu}(x) = \overline{\underline{\lim}}_{\varepsilon\to 0} \frac{\log f_{x,\varepsilon}(\mu)}{\log \varepsilon} = \lim_{s\to\infty} \sup(\inf)_{t\geq s} \frac{\log f_{x,1/t}(\mu)}{-\log t},$$

we set, for each $s \in \mathbb{N}$,

$$\overline{\beta}_{\mu}(x, s) = \sup_{t>s} \frac{\log f_{x,1/t}(\mu)}{-\log t} \quad \text{and} \quad \underline{\beta}_{\mu}(x, s) = \inf_{t>s} \frac{\log f_{x,1/t}(\mu)}{-\log t};$$

note that, for each $x \in X$, $\mathbb{N} \ni s \mapsto \overline{\beta}_\mu(x,s) \in [0,+\infty]$ is non-increasing, whereas $\mathbb{N} \ni s \mapsto \underline{\beta}_\mu(x,s) \in [0,+\infty]$ is a non-decreasing function.

Proposition 1.1. *Let (X,d) be a compact metric space and let $\alpha > 0$. Then, each of the sets*

$$PD = \{\mu \in \mathcal{M}(X) \mid \dim_P^-(\mu) \geq \alpha\},$$

$$HD = \{\mu \in \mathcal{M}(X) \mid \dim_H(\mu) = 0\}$$

is a G_δ subset of $\mathcal{M}(X)$.

Proof. Since the arguments in both proofs are similar, we just prove the statement for PD. We show that $\mathcal{M}(X) \setminus PD$ is an F_σ set.

Claim 1. $PD = \bigcap_{s\in\mathbb{N}}\{\mu \in \mathcal{M}(X) \mid \mu\text{-}\operatorname{ess\,inf} \overline{\beta}_\mu(x,s) \geq \alpha\}$.

Let $\mu \in PD$. Since, for each $x \in X$, $\mathbb{N} \ni s \mapsto \overline{\beta}_\mu(x,s) \in [0,\infty]$ is a non-increasing function, it follows that, for each $s \in \mathbb{N}$, $\mu\text{-}\operatorname{ess\,inf} \overline{\beta}_\mu(x,s) \geq \alpha$.

Now, let $\mu \in \bigcap_{s\in\mathbb{N}}\{\nu \in \mathcal{M}(X) \mid \mu\text{-}\operatorname{ess\,inf} \overline{\beta}_\nu(x,s) \geq \alpha\}$. Then, for each $s \in \mathbb{N}$, there exits a measurable $A_s \subset X$ with $\mu(A_s) = 1$, such that for each $x \in A_s$, $\overline{\beta}_\mu(x,s) \geq \alpha$. Let $A := \bigcap_{s\geq 1} A_s$; then, for each $x \in A$, one has $\overline{d}_\mu(x) \geq \alpha$; given that $\mu(A) = 1$, the result follows by Proposition I.2.

Let $\mu \in \mathcal{M}(X)$, let $k, s \in \mathbb{N}$, set $Z_\mu(s,k) = \{x \in X \mid \overline{\beta}_\mu(x,s) \leq \alpha - 1/k\}$ and set, for each $l \in \mathbb{N}$,

$$\mathcal{M}_{s,k}(l) = \{\nu \in \mathcal{M}(X) \mid \nu(Z_\nu(s,k)) \geq 1/l\}.$$

Claim 2. $Z_\mu(s,k)$ is closed.

Let (z_n) be a sequence in $Z_\mu(s,k)$ such that $z_n \to z$, and let $t > 0$. Since for each fixed $\mu \in \mathcal{M}(X)$, the mapping $X \ni x \mapsto f_{x,1/t}(\mu) \in (0,1]$ is continuous (see the proof of Lemma 1.2), the mapping $X \ni x \longmapsto \overline{\beta}_\mu(x,s) \in [0,+\infty)$ is lower semi-continuous, which implies that $z \in Z_\mu(s,k)$.

Claim 3. $W_{s,k} = \{(\mu, x) \in \mathcal{M}(X) \times X \mid \overline{\beta}_\mu(x,s) > \alpha - 1/k\}$ is open.

This is a consequence of the fact that, by Lemma 1.2, the mapping $\mathcal{M}(X) \times X \ni (\mu, x) \longmapsto \overline{\beta}_\mu(x,s)$ is lower semi-continuous.

Now, we show that $\mathcal{M}_{s,k}(l)$ is closed. Let (μ_n) be a sequence in $\mathcal{M}_{s,k}(l)$ such that $\mu_n \to \mu$. Suppose, by absurd, that $\mu \notin \mathcal{M}_{s,k}(l)$; we will find that $\mu_n \notin \mathcal{M}_{s,k}(l)$ for n sufficiently large, a contradiction.

If $\mu \notin \mathcal{M}_{s,k}(l)$, then $\mu(A) > 1 - 1/l$ where, $A = X \setminus Z_\mu(s,k)$. Given that μ is tight (μ is a probability Borel measure and the space X is Polish), there exists a compact set $C \subset A$ such that $\mu(C) > 1 - 1/l$.

The idea is to construct a suitable subset of $W_{s,k}$ that contains a neighborhood of $\{\mu\} \times C$. Let, for each $x \in C$, $V_x \subset W_{s,k}$ be an open neighborhood of (μ, x) (such open set exists, by Claim 3; that is, $V_x := B((\mu, x); \varepsilon) = \{(\nu, y) \in \mathcal{M}(X) \times X \mid \max\{\rho(\nu, \mu), d(x,y)\} < \varepsilon\}$, for some suitable $\varepsilon > 0$ (where ρ is any metric defined in $\mathcal{M}(X)$ which is compatible with the weak topology); then, $\{V_x\}_{x \in C}$ is an open cover of $\{\mu\} \times C$, and since $\{\mu\} \times C$ is a compact subset of $\mathcal{M}(X) \times X$, it follows that one can extract from $\{V_x\}_{x \in C}$ a finite subcover, $\{V_{x_i}\}_{i=1}^n$.

We affirm that there exists an $\ell \in \mathbb{N}$ (which depends on C) such that $\{\mu_n\}_{n \geq \ell} \subset \bigcap_i (\pi_1(V_{x_i}))$. Namely, for each i, there exists an ℓ_i such that $\{\mu_n\}_{n \geq \ell_i} \subset \pi_1(V_{x_i})$; set $\ell := \max\{\ell_i \mid i \in \{1, \ldots, n\}\}$, and note that for each i, $\{\mu_n\}_{n \geq \ell} \subset \pi_1(V_{x_i})$. Set also $\mathcal{I} := \bigcap_i (\pi_1(V_{x_i}))$ and $\mathcal{O} := \bigcup_i (\pi_2(V_{x_i}))$.

Since for each i, $V_{x_i} = \pi_1(V_{x_i}) \times \pi_2(V_{x_i})$, and given that

$$\begin{aligned}\{\mu_n\}_{n \geq \ell} \times \mathcal{O} \subset \mathcal{I} \times \mathcal{O} = \bigcup_j \left(\left[\bigcap_i \pi_1(V_{x_i}) \right] \times \pi_2(V_{x_j}) \right) \subset \bigcup_j \left(\pi_1(V_{x_j}) \times \pi_2(V_{x_j}) \right) \\ = \bigcup_j V_{x_j} \subset W_{s,k},\end{aligned}$$

it follows that, for each $n \geq \ell$ and each $y \in \mathcal{O}$, $\overline{\beta}_{\mu_n}(y,s) > \alpha - 1/k$. Moreover, $\mathcal{O}$ is an open set that contains C.

On the other hand, weak convergence implies that

$$\limsup_{n\to\infty} \mu_n(X \setminus \mathcal{O}) \leq \mu(X \setminus \mathcal{O}) \leq \mu(X \setminus C) < \frac{1}{l},$$

from which follows that there exists an $\tilde{\ell} \geq \ell$ such that, for $n \geq \tilde{\ell}$, $\mu_n(X \setminus \mathcal{O}) < 1/l$.

Combining the last results, one concludes that for $n \geq \tilde{\ell}$, $\mu_n(X \setminus \mathcal{O}) < 1/l$, and for each $y \in \mathcal{O}$, $\overline{\beta}_{\mu_n}(y,s) > \alpha - 1/k$, so

$$\mu_n(Z_{\mu_n}(s,k)) \leq \mu_n(X \setminus \mathcal{O}) < \frac{1}{l};$$

this contradicts the fact that, for each $n \in \mathbb{N}$, $\mu_n \in \mathcal{M}_{s,k}(l)$. Hence, $\mu \in \mathcal{M}_{s,k}(l)$, and $\mathcal{M}_{s,k}(l)$ is a closed subset of $\mathcal{M}(X)$.

Finally, it follows $\mathcal{M}(X) \setminus PD(\alpha) = \bigcup_{s\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \bigcup_{l\in\mathbb{N}} \mathcal{M}_{s,k}(l)$ is an F_σ subset of $\mathcal{M}(X)$, concluding the proof of the proposition. □

1.1.2 Dense sets

From now on, we assume that (X,d) is a Polish metric space and that $T : X \to X$ is a Lipschitz function, with Lipschitz constant $\Lambda > 1$. Assume also that $T^{-1} : X \to X$ exists as a Lipschitz function, with Lipschitz constant $\Lambda' > 1$.

Proposition 1.2. *Let $\mu \in \mathcal{M}(T)$. Then, for each $x \in X$, $\overline{d}_\mu(x) = \overline{d}_\mu(Tx)$, $\underline{d}_\mu(x) = \underline{d}_\mu(Tx)$.*

Proof. It follows from Birkhoff's Ergodic Theorem that, for each $z \in X$ and each $\varepsilon > 0$, the limit

$$\tilde{\varphi}_{B(z,\varepsilon)}(y) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f_z^\varepsilon(T^i(y)) \tag{1.1}$$

exists for μ-a.e. $y \in X$, and

$$\int \tilde{\varphi}_{B(z,\varepsilon)}(y)d\mu(y) = \int f_z^{\varepsilon}(y)d\mu(y) = f_{z,\varepsilon}(\mu).$$

Fix $x \in \text{supp}(\mu)$. It is straightforward to show that, for each $y \in X$ and each $i \in \mathbb{N} \cup \{0\}$, one has $f_x^{\varepsilon/\Lambda}(T^i(y)) \leq f_{Tx}^{\varepsilon}(T^{i+1}(y))$. Letting $z = x$ and $z = Tx$ in (1.1), respectively, one gets $\tilde{\varphi}_{B(x,\varepsilon/\Lambda)}(y) \leq \tilde{\varphi}_{B(Tx,\varepsilon)}(y)$ for μ-a.e. $y \in X$, from which follows that $f_{x,\varepsilon/\Lambda}(\mu) \leq f_{Tx,\varepsilon}(\mu)$.

Case 1: $x \in \text{supp}(\mu)$. Note that, for each $\eta > 0$, $f_{x,\eta}(\mu) > 0$. Let $\varepsilon = 1/t$, $t = l/\Lambda$ and $s \geq 1 + 1/\Lambda$; then,

$$\begin{aligned}\sup_{t \geq s} \frac{\log f_{Tx,1/t}(\mu)}{-\log t} \quad &\leq \quad \sup_{l \geq \Lambda s} \frac{\log l}{\log l - \log \Lambda} \frac{\log f_{x,1/l}(\mu)}{-\log l} \leq \frac{\log(\Lambda s)}{\log(\Lambda s) - \log \Lambda} \sup_{l \geq \Lambda s} \frac{\log f_{x,1/l}(\mu)}{-\log l} \\ &= \quad A_\Lambda(s) \sup_{l \geq \Lambda s} \frac{\log f_{x,1/l}(\mu)}{-\log l},\end{aligned}$$

where $A_\Lambda(s) := \frac{\log s + \log \Lambda}{\log s}$ (since $s \geq 1 + 1/\Lambda$, one has $l \geq \Lambda + 1$).

Using the same idea, one can prove that $f_{z,\varepsilon/\Lambda'}(\mu) \leq f_{T^{-1}z,\varepsilon}(\mu)$; letting $z = Tx$, one gets $f_{Tx,\varepsilon/\Lambda'}(\mu) \leq f_{x,\varepsilon}(\mu)$. Thus, the previous discussion leads to

$$\overline{\beta}_\mu(Tx, s) \leq A_\Lambda(s)\overline{\beta}_\mu(x, \Lambda s) \quad \text{and} \quad \overline{\beta}_\mu(x, s) \leq A_{\Lambda'}(s)\overline{\beta}_\mu(Tx, \Lambda' s);$$

one can combine these inequalities and obtain, for each $x \in X$ and each $s \geq \max\{1 + 1/\Lambda, 1 + 1/\Lambda'\}$,

$$\overline{\beta}_\mu(Tx, s) \leq A_\Lambda(s)\overline{\beta}_\mu(x, \Lambda s) \leq A_\Lambda(s)A_{\Lambda'}(\Lambda s)\overline{\beta}_\mu(Tx, \Lambda \cdot \Lambda' s).$$

Now, taking the limit $s \to \infty$ in the inequalities above and observing that $A_\Lambda(s)$ and $A_{\Lambda'}(s)$ are decreasing functions such that $\lim_{s\to\infty} A_{\Lambda(\Lambda')}(s) = 1$, one gets $\overline{d}_\mu(Tx) = \overline{d}_\mu(x)$.

Case 2: $x \notin \text{supp}(\mu)$. It follows from the T-invariance of $\text{supp}(\mu)$ that $T(x) \notin \text{supp}(\mu)$; thus, $\overline{d}_\mu(Tx) = +\infty = \overline{d}_\mu(x)$.

The proof that, for each $x \in X$ $\underline{d}_\mu(Tx) = \underline{d}_\mu(x)$, is analogous; therefore, we omit it. $\square$

Remark 1.3. If $\mu \in \mathcal{M}_e$, since T and T^{-1} are Lipschitz functions, it follows from Proposition 1.2 that $\overline{d}_\mu(x)$ and $\underline{d}_\mu(x)$ are constants μ-a.e. (an analogous result can be found in [13], Theorem 4.1.10 chapter 1).

Lemma 1.3. *Let X be a compact metric space. If $\mu \in \mathcal{M}(T)$, then* $\dim_P^+(\mu) \geq \frac{h_\mu(T)}{\log \Lambda}$.

Proof. Fix $x \in X$, $n \geq 1$ and $\varepsilon > 0$. Given $y \in B(x, \varepsilon\Lambda^{-n})$, one has, for each $0 \leq i \leq n$, $\rho(T^i y, T^i x) \leq \Lambda^i \rho(x,y) \leq \Lambda^{i-n}\varepsilon < \varepsilon$, which shows that $y \in B(x,n,\varepsilon) := \{z \in X \mid \rho(T^i z, T^i x) < \varepsilon,\ \forall\, 0 \leq i \leq n-1\}$. Hence, for each $x \in X$ and each $\varepsilon > 0$,

$$\begin{aligned}\overline{d}_\mu(x) \geq \limsup_{n\to\infty} \frac{\log \mu(B(x,\varepsilon\Lambda^{-n}))}{\log \varepsilon\Lambda^{-n}} \quad &\geq \quad \limsup_{n\to\infty} \frac{\log \mu(B(x,n,\varepsilon))}{-n} \frac{1}{\frac{-\log\varepsilon}{n} + \log\Lambda} \\ &\geq \quad \limsup_{n\to\infty} \frac{\log \mu(B(x,n,\varepsilon))}{-n} \frac{1}{\log \Lambda};\end{aligned}$$

it follows that, for μ-a.e. $x \in X$,

$$\begin{aligned}\overline{d}_\mu(x) \geq \lim_{\varepsilon\to 0}\limsup_{n\to\infty} \frac{\log \mu(B(y,\varepsilon\Lambda^{-n}))}{\log \varepsilon\Lambda^{-n}} \quad &\geq \quad \lim_{\varepsilon\to 0}\limsup_{n\to\infty} \frac{\log \mu(B(x,n,\varepsilon))}{-n} \frac{1}{\log \Lambda} \\ &= \quad h_\mu(T,x)\frac{1}{\log\Lambda},\end{aligned} \tag{1.2}$$

where $h_\mu(T,x)$ is the so-called local entropy of μ at x. One also has, using Brin-Katok Theorem, that $\int h_\mu(T,x)d\mu(x) = h_\mu(T)$ (the compactness of X is required in this step; see [10]). Hence, there exists a measurable set B, with $\mu(B) > 0$, such that, for each $x \in B$, $\overline{d}_\mu(x) \geq \frac{h_\mu(T)}{\log\Lambda}$. The result is now a consequence of Proposition I.2. $\square$

Lemma 1.4. *Let X be a Polish metric space and let $\mu \in \mathcal{M}_e$. Then,* $\dim_P^-(\mu) \geq \frac{h_\mu(T)}{\log \Lambda}$.

Proof. Since μ is ergodic, it follows from Proposition 1.2 that $\overline{d}_\mu(x)$ is constant for μ-a.e. x (this constant may be infinite).

One also has, by Lemma C.1, that $\int \underline{h}_\mu(T,x)d\mu(x) = \mu\text{-}\operatorname{ess\,inf} \underline{h}_\mu(T,x)$, and then, by Theorem C.2, that $\int \underline{h}_\mu(T,x)d\mu(x) \geq h_\mu(T)$. Thus, by inequality (1.2), one gets, for

μ-a.e. $x \in X$,

$$\overline{d}_\mu(x) = \int \overline{d}_\mu(x) d\mu(x) \geq \int \underline{h}_\mu(T, x) \frac{1}{\log \Lambda} d\mu(x) \geq h_\mu(T) \frac{1}{\log \Lambda}.$$

The result is obtained again by an application of Proposition I.2. □

Lemma 1.5 (Lemma 6 in [51]). *Let $\mu \in \overline{\mathcal{M}}(T)$ be such that $\mu(\mathbb{R}^{\mathbb{Z}}) = 1$, and let $s_0 > 0$. Then, μ can be approximated by a T-periodic measure $\mu_x \in \overline{\mathcal{M}}(T)$, where $x = (x_i)$ is in $\mathbb{R}^{\mathbb{Z}}$, x has period $s \geq s_0$ and $x_i \neq x_j$ if $i \neq j$, $i, j = 1, \ldots, s$.*

The next result is an extension of Lemma 1.5 to $X = \prod_{-\infty}^{+\infty} M$, where M is perfect and compact (the hypothesis of M being perfect is required to guarantee that one can always choose the periodic point x in the statement of Lemma 1.6 in such way that $x_i \neq x_j$ if $i \neq j$, $i, j = 1, \ldots, s$; see Remark 3.4 in [38]). Lemma 2.3 of Chapter II is a weak version of Lemma 1.6, actually it is sufficient (see Section 2.2 of Chapter II for a proof of Lemma 2.3).

Lemma 1.6. *Let $X = \prod_{-\infty}^{+\infty} M$, where M is perfect and compact, let $\mu \in \mathcal{M}(T)$ and let $s_0 > 0$. Then, μ can be approximated by a T-periodic measure $\mu_x \in \mathcal{M}(T)$ such that $x \in X$ has period $s \geq s_0$ and $x_i \neq x_j$ if $i \neq j$, $i, j = 1, \ldots, s$.*

Remark 1.4. Since, for each $x \in X$, $\mu_x(\cdot) = \frac{1}{k_x} \sum_{i=0}^{k_x - 1} \delta_{T^i x}(\cdot)$, where k_x is the period of x, the measure presented in the statement of Lemma 1.6 is clearly supported on X, so it belongs to $\mathcal{M}(T)$.

Lemma 1.7 (Lemma 7 in [51]). *Let $\mu \in \overline{\mathcal{M}}(T)$ be such that $\mu(\mathbb{R}^{\mathbb{Z}}) = 1$, and let $K > 0$. Then, every neighborhood V of μ contains a ρ, with $\rho(X) = 1$, such that $h_\rho(T) > K$.*

Proposition 1.3 (Proposition 6.1 in [38]). *Let $\mu_n \to \mu$ in the space of all normalized Borel measures in a compact metric space Y. Let E be a Borel subset of Y such that $\mu_n(E) = 1$ for all $n \geq 0$. Then, $\int f d\mu_n \to \int f d\mu$ for any bounded Borel measurable function f on Y such that $f|_E$ is continuous.*

The next result is an extension of Lemma 1.7 to the space $X = \prod_{-\infty}^{+\infty} M$ (with M a perfect Polish space) proved using Lemma 1.6 and Proposition 1.3. We leave the details for the dedicated reader.

Proposition 1.4. *Let M a perfect Polish space, $\mu \in \mathcal{M}(T)$ and let $K > 0$. Then, every neighborhood V of μ contains an invariant measure $\rho \in \mathcal{M}(T)$ such that $h_\rho(T) \geq K$.*

Proof. Take $V_\mu(f_1, \cdots, f_d; \delta)$, L and $\kappa > 0$ as in Proposition 2.7. The proof is analogous to proof of Proposition 2.7, with the difference that one should take $s_0 > 0$ such that

$$\kappa \log(s_0 - 1) - \kappa \log \kappa - (1 - \kappa) \log(1 - \kappa) \geq K.$$

It follows, as in Proposition 2.7, from Lemma 1.6 that there exists a T-periodic point $w = (w_i) \in M^{\mathbb{Z}}$, with period $s \geq s_0$, such that $w_i \neq w_j$ for each $i \neq j$, $i, j = 1, \dots, s$, and $\mu_w \in V_\mu(f_1, \cdots, f_d; \delta/2)$.

Following the proof of Lemma 7 in [51], one defines, for each fixed $s \geq s_0$, a Markov chain ρ whose states are $w_1, \dots, w_s$, whose initial probabilities are given by the s-tuple $(1/s, \dots, 1/s)$, and whose transition probabilities are given by the $s \times s$-matrix p_{ij}, where

$$\begin{aligned} p_{s\,1} &= 1 - \kappa, \\ p_{i\,i+1} &= 1 - \kappa, \quad \text{for } i = 1, \dots, s-1, \\ p_{i\,j} &= \frac{\kappa}{s-1}, \quad \text{otherwise.} \end{aligned}$$

The entropy for a stochastic process given by a Markov chain with initial probabilities π_i $(1 \leq i \leq s)$ and transition matrix p_{ij} $(1 \leq i, j \leq s)$ is given by $\sum_{i,j} \pi_i p_{ij} \cdot \log p_{ij}$. Therefore, one obtains

$$h_\rho(T) \geq \kappa \log(s_0 - 1) - \kappa \log \kappa - (1 - \kappa) \log(1 - \kappa) \geq K.$$

To evaluate the integral $\left| \int f_j d\mu_w - \int f_j d\rho \right|$, we proceed as in the Proposition 2.7 in order to show that $\rho \in V_{\mu_w}(f_1, \cdots, f_d; \delta/2)$, from which follows that $\rho \in V_\mu(f_1, \cdots, f_d; \delta)$. □

Proposition 1.5. *Let $L > 0$. Then, $\{\mu \in \mathcal{M}_e \mid \dim_P^-(\mu) \geq L\}$ is a dense subset of $\mathcal{M}_e$.*

Proof. Let $\delta > 0$, and let $\mu \in \mathcal{M}_e$. It is straightforward to show that T is a Lipschitz function with constant $\Lambda = 2$. Set $K := L \log 2$. It follows from the proof of Proposition 1.4

(see Lemma 7 in [51]) that given any neighborhood of μ (in the induced topology) of the form $V_\mu(f_1,\ldots,f_r;\delta)=\{\nu\in\mathcal{M}_e \mid |\int f_i\,d\nu-\int f_i\,d\mu|<\delta,\ i=1,\ldots,r\}$ (where $\delta>0$ and each $f_i:X\to\mathbb{C}$ is continuous and bounded; such sets form a sub-basis of the weak topology), there exists a measure $\zeta\in V_\mu(f_1,\ldots,f_r;\delta)$ such that $h_\zeta(T)\geq K$. Now, by Lemma 1.4, one has $\dim_P^-(\zeta)\geq \frac{h_\zeta(T)}{\log 2}=K\frac{1}{\log 2}=L$. □

Proposition 1.6. *Let M be a perfect and separable metric space. Then, the set $\{\mu\in\mathcal{M}_e\mid \dim_H^+(\mu)=0\}$ is dense in $\mathcal{M}_e$.*

Proof. Let μ be the T-periodic measure associated with the T-period point $x\in X$, and denote its period by k. Naturally, $\mu(\cdot)=\frac{1}{k}\sum_{i=0}^{k-1}\delta_{f^i(x)}(\cdot)$, and for each $i=0,\cdots,k-1$, one has

$$\overline{d}_\mu(T^i(x))=\limsup_{r\to 0}\frac{\log\mu(B(T^i(x),r))}{\log r}=\limsup_{r\to 0}\frac{-\log k}{\log r}=0.$$

The result follows now from the fact that the set of T-periodic measures is dense in $\mathcal{M}_e$ (see Theorem 3.3 in [39]). □

Remark 1.5. The result stated in Proposition 1.6 is valid for any topological dynamical system (X,T) such that the set of T-periodic measures is dense in $\mathcal{M}_e$; this is particularly true for systems which satisfy the specification property (see Remark 1.2 for more details).

Proof **(Theorem 1.1)**.

III. Note that, by Propositions 1.1, 1.5 and item I of Theorem 1.1, $PD=\bigcup_{L\geq 1}PD(L)$ is a countable intersection of dense G_δ subsets of $\mathcal{M}(T)$.

IV. It follows from Propositions 1.1 and 1.6 that HD is a dense G_δ subset of $\mathcal{M}_e$. The result is now a consequence of item I in Theorem 1.1.

□

The next statement says that each $\mu\in HD\cap PD\cap C_X$ is supported on a dense G_δ subset of X.

Proposition 1.7. *Let $\mu \in PD \cap HD \cap C_X$. Then, each of the sets $\overline{\mathfrak{D}}_\mu = \{x \in X \mid \overline{d}_\mu(x) = \infty\}$ and $\underline{\mathfrak{D}}_\mu = \{x \in X \mid \underline{d}_\mu(x) = 0\}$ is a dense G_δ subset of X.*

Proof. We just present the proof that $\overline{\mathfrak{D}}_\mu$ is a dense G_δ subset of X.

$\overline{\mathfrak{D}}_\mu$ *is a G_δ set in X.* Let $\alpha > 0$, let $s \in \mathbb{N}$, and set $Z_{\mu,s}(\alpha) = \{x \in X \mid \overline{\beta_\mu}(x, s) \leq \alpha\}$. Following the proof of Claim 2 in Proposition 1.1, it is clear that $Z_{\mu,s}(\alpha)$ is closed. Thus, taking $\alpha = n \in \mathbb{N}$, it follows that $\overline{\mathfrak{D}}_\mu = \bigcap_{n\in\mathbb{N}} \bigcap_{s\in\mathbb{N}} (X \setminus Z_{\mu,s}(n))$ is a G_δ subset of X.

$\overline{\mathfrak{D}}_\mu$ *is dense in X.* Since $\mu \in PD$, one has $\mu(\overline{\mathfrak{D}_\mu}) = 1$. Suppose that $\overline{\mathfrak{D}}_\mu$ is not dense; then, there exist $x \in X$ and $\varepsilon > 0$ such that $B(x, \varepsilon) \cap \overline{\mathfrak{D}}_\mu = \emptyset$. This implies that $1 = \mu(\overline{\mathfrak{D}}_\mu) + \mu(B(x, \varepsilon))$, which is an absurd, since $\mu(B(x, \varepsilon)) > 0$ (recall that $\mathrm{supp}(\mu) = X$, given that $\mu \in C_X$). □

1.2 Rates of recurrence and quantitative waiting time indicators almost everywhere

In this section, we present the proof of items V-VIII of Theorem 1.1. This section presents the counterparts, for $\overline{R}(x)$, $\underline{R}(x)$, $\overline{R}(x, y)$ and $\underline{R}(x, y)$, of the results presented in Section 1.1. Once again, we assume that (X, d) is a Polish metric space and that $T : X \to X$ is a Lipschitz function, with Lipschitz constant $\Lambda > 1$. Assume also that $T^{-1} : X \to X$ exists as a Lipschitz function, with Lipschitz constant $\Lambda' > 1$.

1.2.1 Sets of ergodic measures with zero lower and infinity upper rates of recurrence

Proposition 1.8. *Let (X, T) be a topological dynamical system and let $\alpha > 0$. Then, each of the sets*

$$\begin{aligned} \overline{\mathcal{R}} &= \{\mu \in \mathcal{M}(T) \mid \mu\text{-}\operatorname{ess\,inf} \overline{R}(x) \geq \alpha\}, \\ \underline{\mathcal{R}} &= \{\mu \in \mathcal{M}(T) \mid \mu\text{-}\operatorname{ess\,sup} \underline{R}(x) = 0\} \end{aligned}$$

is G_δ subset of $\mathcal{M}(T)$.

Proof. Since the arguments in both proofs are similar, we just prove the statement for $\overline{\mathcal{R}}$. We show that $\mathcal{M}(T) \setminus \overline{\mathcal{R}}$ is an F_σ set. We begin noting that $\overline{\mathcal{R}} = \bigcap_{s\in\mathbb{N}}\{\mu \in \mathcal{M}(T) \mid \mu\text{-}\operatorname{ess\,inf} \overline{\gamma}(x,s) \geq \alpha\}$ (see the proof of Proposition 1.1 for the proof of an analogous statement).

Let $l, s \in \mathbb{N}$, set $Z_{s,l} = \{x \in X \mid \overline{\gamma}(x,s) \leq \alpha - 1/l\}$, and set for each $k \in \mathbb{N}$,

$$\mathcal{M}_{s,l}(k) = \{\mu \in \mathcal{M}(T) \mid \mu(Z_{s,l}) \geq 1/k\}.$$

Claim. $Z_{s,l}$ is closed.

Let (z_n) be a sequence in $Z_{s,l}$ such that $z_n \to z$. Since, for each $n \in \mathbb{N}$, $\overline{\gamma}(z_n, s) = \sup_{s\geq t} \frac{\log \tau_{1/t}(z_n)}{\log t} \leq \alpha - 1/l$, it follows that for each $t \geq s$, $\tau_{1/t}(z_n) \leq t^{\alpha-1/l}$. Now, fix $t \geq s$; then, there exists a sequence (k_n), $k_n \in \mathbb{N}$, such that for each $n \in \mathbb{N}$, $k_n \leq t^{\alpha-1/l}$ and $d(T^{k_n}(z_n), z_n) \leq 1/t$.

Given that (k_n) is bounded, there exist a sub-sequence (k_{n_j}), $k \leq t^{\alpha-1/l}$ and $j_0 \in \mathbb{N}$ such that for each $j \geq j_0$, $k_{n_j} = k$. Since for each $j \in \mathbb{N}$, $d(T^{k_{n_j}}(z_{n_j}), z_{n_j}) \leq 1/t$, it follows from the continuity of T^k and the previous statements that $d(T^k(z), z) \leq 1/t$, which proves that $\tau_{1/t}(z) \leq t^{\alpha-1/l}$. Given that $t \leq s$ is arbitrary, one gets $\sup_{t\geq s} \frac{\log \tau_{1/t}(z)}{\log t} \leq \alpha - 1/l$, which concludes that $z \in Z_{s,l}$.

Now, we show that $\mathcal{M}_{s,l}(k)$ is closed. Indeed, fix $s \in \mathbb{N}$ and let (μ_n) be a sequence in $\mathcal{M}_{s,l}(k)$ such that $\mu_n \to \mu$. Suppose that $\mu \notin \mathcal{M}_{s,l}(k)$; then, $\mu(A) > 1 - 1/k$, where $A = X \setminus Z_{s,l}$. Since, by Claim, A is open in X and $\mu_n \to \mu$, it follows that $\liminf_{n\to\infty} \mu_n(A) \geq \mu(A) > 1 - 1/k$, which shows that there exists an $\ell \in \mathbb{N}$ such that $\mu_\ell(A) > 1 - 1/k$. This contradicts the fact that $\mu_\ell \in \mathcal{M}_{s,l}(k)$. Hence, $\mu \in \mathcal{M}_{s,l}(k)$.

Given that $\mathcal{M}_{s,l}(k)$ is closed, it follows that $\mathcal{M}(T)\setminus\overline{\mathcal{R}}(\alpha) = \bigcup_{s\in\mathbb{N}}\bigcup_{k\in\mathbb{N}}\bigcup_{l\in\mathbb{N}} \mathcal{M}_{s,l}(k) = \{\mu \in \mathcal{M}(T) \mid \mu\text{-}\operatorname{ess\,inf} \overline{\gamma}_\mu(x,s) < \alpha\}$ is an F_σ subset of $\mathcal{M}(T)$. $\square$

The next results show that such sets are dense for the full-shift system.

Proposition 1.9. *Let (X,T) be the full-shift system, with M a Polish metric space. Then, $\underline{\mathcal{R}} = \{\mu \in \mathcal{M}_e \mid \mu\text{-}\operatorname{ess\,sup} \underline{R}(x) = 0\}$ is a dense subset of $\mathcal{M}_e$.*

Proof. Note that if μ_x is a T-periodic measure, then for each $y \in \mathcal{O}(x)$, $R(y) = 0$. The result follows, therefore, from the fact that the set of T-periodic measures is dense in $\mathcal{M}_e$ (this is Theorem 3.3 in [39]). □

Proposition 1.10. *Let (X,T) be the full-shift system, with M a perfect and compact metric space, and let $L > 0$. Then, $\overline{\mathcal{R}}(L) = \{\mu \in \mathcal{M}_e \mid \mu\text{-}\operatorname{ess\,inf} \overline{R}(x) \geq L\}$ is a dense subset of $\mathcal{M}(T)$.*

Proof. Fix $x \in X$, $n \geq 1$ and $\varepsilon > 0$. It follows from the argument presented in the proof of Lemma 1.4 that $B(x, \varepsilon 2^{-n}) \subset B(x,n,\varepsilon)$ (recall that the full-shift system is Lipschitz continuous). Note that $\tau_{\varepsilon 2^{-n}}(x) \geq R_n(x,\varepsilon)$, where $R_n(x,\varepsilon) = \inf\{k \geq 1 \mid T^k(x) \in B(x,n,\varepsilon)\}$ is the nth return time to the dynamical ball $B(x,n,\varepsilon)$. Now, as in the proof of Proposition 1.5, for each $\mu \in \mathcal{M}(T)$ and each neighborhood $V_\mu(f_1,\dots,f_n;\delta)$ (in the induced topology), there exists a measure $\zeta \in V_\mu(f_1,\dots,f_n;\delta) \cap \mathcal{M}_e$ such that $h_\zeta(T) \geq L \log 2$. The result is now a consequence of Theorem C.3 and Proposition C.1, which state that $\underline{R}(x) \geq \frac{h_\zeta(T)}{\log 2} = L$, for ζ-a.e. x. □

Remark 1.6. Proposition 1.10 can be extended to the case where M is a Polish metric space using an adapted version of Lemma 1.4. Namely, let $\mu \in \mathcal{M}_e$. It follows from the proof of Theorem A in [56] that $\underline{h}(T,x) := \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log R_n(x,\varepsilon)$ is T-invariant (and therefore, constant for μ-a.e. x; this constant may be infinite), where $R_n(x,\varepsilon) := \inf\{k \in \mathbb{N} \mid f^k(x) \in B(x,n,\varepsilon)\}$, and that $h_\mu(T) \leq \underline{h}(T,x)$ for μ-a.e. x; as in the proof of Katok's Theorem, this inequality is also valid for Polish spaces (see the discussion preceding Theorem 2.6 in [44]).

Since $R(x)$ is also T-invariant, it follows from the previous discussion and from the argument presented in the proof of Proposition A in [56] that, for μ-a.e. $x \in X$,

$$\underline{R}(x) = \int \underline{R}(x) d\mu(x) \geq \int \frac{\underline{h}_\mu(T,x)}{\log 2} d\mu(x) \geq \frac{h_\mu(T)}{\log 2}.$$

***Proof* (Theorem 1.1).**

V. The result is a consequence of Propositions 1.8, 1.9 and item I of Theorem 1.1.

VI. It follows from Proposition 1.8, Remark 1.6 and item I of Theorem 1.1, since $\overline{\mathcal{R}} = \bigcap_{L\geq 1} \overline{\mathcal{R}}(L)$.

□

Remark 1.7. There is an alternative proof to the fact that $\underline{\mathcal{R}} = \{\mu \in \mathcal{M}_e \mid \mu\text{-}\operatorname{ess\,inf} \underline{R}(x) = 0\}$ is residual in $\mathcal{M}(T)$. In fact, this result can be seen as a direct consequence of Theorem 2 in [6] and Theorem 1.1-III, since it follows that, for each $\mu \in HD$, $\mu\text{-}\operatorname{ess\,sup} \underline{R}(x) \leq \dim_H(\mu) = 0$.

The following result states that each typical measure obtained in Theorem 1.1 is supported on the dense G_δ set $\mathfrak{R} = \{x \in X \mid \underline{R}(x) = 0 \text{ and } \overline{R}(x) = \infty\}$.

Proposition 1.11. *Let (X,T) the full-shift system, where M is a perfect and separable metric space. Then, each of the sets $\mathfrak{R}^- = \{x \in X \mid \overline{R}(x) = \infty\}$ and $\mathfrak{R}_- = \{x \in X \mid \underline{R}(x) = 0\}$ is a dense G_δ subset of X. Moreover, for each $\mu \in \overline{\mathcal{R}} \cap \underline{\mathcal{R}} \cap C_X$, $\mu(\mathfrak{R}^- \cap \mathfrak{R}_-) = 1$.*

Proof. We just present the proof that $\mathfrak{R}^-$ is a dense G_δ subset of X.

$\mathfrak{R}^-$ is a G_δ set in X. Let $\alpha > 0$, $s \in \mathbb{N}$, and set $Z_s(\alpha) = \{x \in X \mid \overline{\gamma}(x,s) \leq \alpha\}$. Following the proof of Claim in Proposition 1.8, it is clear that $Z_s(\alpha)$ is closed. Thus, taking $\alpha = n \in \mathbb{N}$, it follows that $\mathfrak{R}^- = \bigcap_{n\in\mathbb{N}} \bigcap_{s\in\mathbb{N}} (X \setminus Z_s(n))$ is a G_δ set in X.

$\mathfrak{R}^-$ is dense in X. Let $\mu \in \overline{\mathcal{R}} \cap C_X$. Then, $\mu(\mathfrak{R}^-) = 1$. Suppose that $\mathfrak{R}^-$ is not dense; so, there exist $x \in X$ and $\varepsilon > 0$ such that $B(x,\varepsilon) \cap \mathfrak{R}^- = \emptyset$. This implies that $1 = \mu(\mathfrak{R}^-) + \mu(B(x,\varepsilon))$, which is an absurd, since $\mu(B(x,\varepsilon)) > 0$. □

1.2.2 Sets of ergodic measures with zero lower and infinity upper quantitative waiting time indicators almost everywhere

Now, we present equivalent results, to those already obtained in this section, for the quantitative waiting time indicators.

Let, for each $(x,y) \in X \times X$ and each $s \in \mathbb{N}$, $\overline{\gamma}(x,y,s) := \sup_{t>s} \frac{\log \tau_{1/t}(x,y)}{\log t}$ and $\underline{\gamma}(x,y,s) := \inf_{t>s} \frac{\log \tau_{1/t}(x,y)}{\log t}$.

Proposition 1.12. *Let (X,T) be a topological dynamical system and let $\alpha > 0$. Then, the set*

$$\underline{\mathscr{R}}_\alpha \quad = \quad \{\mu \in \mathcal{M}(T) \mid (\mu \times \mu)\text{-}\operatorname{ess\,sup} \underline{R}(x,y) \leq \alpha\}$$

is a G_δ subset of $\mathcal{M}(T)$.

Proof. Using the same ideas presented in the proof of the Proposition 1.8, we show that $\underline{\mathscr{R}}_\alpha$ is a G_δ subset of $\mathcal{M}(T)$ by showing that $\mathcal{M}(T) \setminus \underline{\mathscr{R}}_\alpha = \bigcup_{s\in\mathbb{N}}\{\mu \in \mathcal{M}(T) \mid (\mu \times \mu)\text{-}\operatorname{ess\,sup} \underline{\gamma}_\mu(x,y,s) > \alpha\}$ is an F_σ set.

Let $l, s \in \mathbb{N}$, set $Z_{s,l} = \{(x,y) \in X \times X \mid \underline{\gamma}(x,y,s) \geq \alpha + 1/l\}$ and set, for each $k \in \mathbb{N}$,

$$\mathcal{M}_{s,l}(k) = \{\mu \in \mathcal{M}(T) \mid (\mu \times \mu)(Z_{s,l}) \geq 1/k\}.$$

The proofs that $Z_{s,l}$ and $\mathcal{M}_{s,l}(k)$ are closed sets follow the same arguments presented in the proof of Proposition 1.8 (here, one uses Theorem 8.4.10 in [8] for the product measure $\mu \times \mu$). Finally, since $\mathcal{M}(T) \setminus \underline{\mathscr{R}}(\alpha) = \bigcup_{s\in\mathbb{N}} \bigcup_{l\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \mathcal{M}_{s,l}(k)$, we are done. □

Proof **(Theorem 1.1)**.

VII. Since, by Proposition 1.12, $\underline{\mathscr{R}} = \{\mu \in \mathcal{M}_e \mid (\mu\times\mu)\text{-}\operatorname{ess\,sup} \underline{R}(x,y) = 0\} = \cap_{k\geq1}\underline{\mathscr{R}}_{1/k}$, one just needs to prove that $\underline{\mathscr{R}}$ is dense. Let μ_z be a T-periodic measure. Then, for each $x, y \in \mathcal{O}(z)$, $R(x, y) = 0$. The result follows from the fact that the set of T-periodic measures is dense in $\mathcal{M}_e$ and from item I in Theorem 1.1.

VIII. The result is a direct consequence of Theorem 1.1 (IV) and the second inequality in (3) (see Theorem 4 in [20]).

$\square$

Proposition 1.13. *Let (X, T) the full-shift system, where M is a perfect and separable metric space. Then, each of the sets $\mathfrak{S}^- = \{(x, y) \in X \times X \mid \overline{R}(x, y) = \infty\}$ and $\mathfrak{S}_- = \{(x, y) \in X \times X \mid \underline{R}(x, y) = 0\}$ is a dense G_δ subset of $X \times X$. Moreover, for each $\mu \in \overline{\mathscr{R}} \cap \underline{\mathscr{R}} \cap C_X$, $(\mu \times \mu)(\mathfrak{S}^- \cap \mathfrak{S}_-) = 1$.*

CHAPTER II

GENERALIZED FRACTAL DIMENSIONS OF INVARIANT MEASURES OF FULL-SHIFT SYSTEMS OVER UNCOUNTABLE ALPHABETS: GENERIC BEHAVIOR

In this chapter, we are interested in extending the analysis of the previous chapter to the so-called upper and lower q-generalized fractal dimensions of the T-invariant measures, $D_q^{\pm}(\mu)$, with $q > 0$ (Definition I.16). We shall also explore the connection between such properties and the orbital behavior of the full-shift system through the upper and lower q-correlation dimensions at a point $x \in X$, for $q \in \mathbb{N} \setminus \{1\}$ (Definition (4)). All results in this Chapter appear in our article published in Forum Mathematicum,V33, N2, p. 435-450 ([12], in January 2021).

As in Chapter I, we begin this chapter making some comments and observations about the results obtained here, as well as some of their dynamical and topological consequences. The proofs of the main results, stated in Theorems 2.1 and 2.2, are presented in Section 2.1, Section 2.2 and Section 2.3.

Our first central result (in this chapter) establishes that if $\mathcal{M}_p(T)$ (the set of T-periodic measures) is dense in $\mathcal{M}(T)$, then generically, for each $s \in (0,1)$, $\mu \in \mathcal{M}(T)$ has s-lower generalized fractal dimension equal to zero. This density is particularly true for dynamical systems satisfying the specification property (such as Axiom A systems [53] and the actions of discrete countable residually finite amenable groups on compact metric spaces with specification property [43]), as previously discussed.

Here, we let (X, d) be a compact metric space.

Theorem 2.1. *Let (X, T) be a topological dynamical system and suppose that $\mathcal{M}_p(T)$ is dense in $\mathcal{M}(T)$. Then, for each $s \in (0, 1)$,*

$$FD = \{\mu \in \mathcal{M}(T) \mid D_\mu^-(s) = 0\}$$

is a residual subset of $\mathcal{M}(T)$.

The next result is a direct consequence of Theorem 2.1 and Proposition I.2.

Corollary 2.1. *Let (X, T) be a topological dynamical system and suppose that $\mathcal{M}_p(T)$ is dense in $\mathcal{M}(T)$. Then,*

$$HD = \{\mu \in \mathcal{M}(T) \mid \dim_H^+(\mu) = 0\}$$

is a residual subset of $\mathcal{M}(T)$.

The first consequence of Corollary 2.1 is that a typical invariant measure is supported on a set $Z \subset X$ satisfying $\dim_H(Z) = 0$; moreover, given that $\dim_H(Z) \geq \dim_{top}(Z)$, it follows that Z is totally disconnected. Now, if (X, T) satisfies the specification property, it is known that $C_X(T)$, the set of invariant measures with $\mathrm{supp}(\mu) = X$, is a dense G_δ subset of $\mathcal{M}(T)$ (see [15, 53]); thus, in this case, Z is a totally disconnected and dense subset of X.

One must compare Corollary 2.1 with Theorem 1.1; although $X = \prod_{-\infty}^{+\infty} M$ may not be compact in Theorem 1.1 (III), X must be endowed with a metric such that T and T^{-1} are both Lipschitz (here, it is only required that the induced topology and the product topology are compatible).

Corollary 2.2. *Let (X, T) be a topological dynamical system and suppose that $\mathcal{M}_p(T)$ is dense in $\mathcal{M}(T)$. Then,*

$$\underline{\mathcal{R}} = \{\mu \in \mathcal{M}(T) \mid \underline{R}(x) = 0, \text{ for } \mu\text{-a.e. } x\}$$

is a residual subset of $\mathcal{M}(T)$.

As before, one may establish the same kind of comparison between Corollary 2.2 and Theorem 1.1-(V): here, it is required that X is compact (there, it is sufficient for X to be Polish); here, the metric may be any one compatible with the product topology (there, it must be such that T and T^{-1} are both Lipschitz).

Returning to the full-shift system, we consider now the case where X is perfect and compact, and two different settings: X is endowed with any metric compatible with the product topology, or it is endowed with a sub-exponential metric of the form

$$r(x,y) = \sum_{|n|\geq 0} \min\left\{\frac{1}{a_{|n|}+1}, d(x_n,y_n)\right\}, \tag{2.1}$$

where $x = (\ldots, x_{-n}, \ldots, x_n, \ldots), y = (\ldots, y_{-n}, \ldots, y_n, \ldots)$, with (a_n) any monotone increasing sequence such that $\sum_{k\geq 0} \frac{1}{a_k+1} < \infty$ and, for each $\alpha > 0$, $\lim_{k\to\infty} \frac{a_k}{e^{\alpha k}} = 0$ (for instance, let for each $n \in \mathbb{N}\cup\{0\}$, $a_n = n^2$); naturally, these metrics induce topologies in X which also are compatible with the product topology.

Our second central result is stated in the following theorem.

Theorem 2.2. *Let $q > 1$. Then,*

$$CD = \{\mu \in \mathcal{M}(T) \mid D^+_\mu(q) = +\infty\}$$

is a dense G_δ subset of $\mathcal{M}(T)$.

Theorems 1.1, 2.1 and 2.2 may be combined with Proposition I.2 in order to produce the following result. Let $q \in \mathbb{N}\setminus\{1\}$; if $\mu \in FD\cap CD$, then there exists a Borel set $Z \subset X$, $\mu(Z) = 1$, such that for each $x \in Z$, one has $\underline{\alpha}_q(x) = D^-_\mu(q) = 0$ and $\overline{\alpha}_q(x) = D^+_\mu(q) = \infty$.

Let $x \in Z$; since $\underline{\alpha}_q(x) = 0$, it follows that given $0 < \alpha \ll 1$ and $R > 0$, there exist a radial sequence (ε_k), with $\varepsilon_k \in (0,R)$, and an $N_k = N_k(x,\alpha,R) \in \mathbb{N}$ such that, for each $n > N_k$, one has $C_q(x,n,\varepsilon_k) \geq \varepsilon_k^{(q-1)\alpha}$. Thus, there exists a scale (defined by (ε_k)) such that $F_k = \operatorname{card}\{(i_1\cdots i_q) \in \{0,1,\cdots,n\}^q \mid r(T^{i_j}(x), T^{i_l}(x)) \leq \varepsilon_k$ for each $0 \leq j,l \leq q\} \geq \varepsilon_k^{(q-1)\alpha}\, n^q$; in this scale, the quantity F_k is of order n^q for each n and each k large enough. This means that, at least in this scale, the orbit of a typical point (with respect to μ) is very "tight" (it is some sense, similar to a periodic orbit).

Nonetheless, since $\overline{\alpha}_q(x) = +\infty$, it follows that given $\beta \gg 1$ and $S > 0$, there exist a radial sequence (s_ℓ), with $s_\ell \in (0, S)$, and an $N_\ell \in \mathbb{N}$ such that, for each $n > N_\ell$, one has $C_q(x, n, s_\ell) \leq s_\ell^{(q-1)\beta}$. Thus, there exists a scale such that

$$\begin{aligned} P_\ell &= \text{card}\,\{(i_1 \cdots i_q) \in \{0, 1, \cdots, n\}^q \mid r(f^{i_j}(x), f^{i_l}(x)) \leq s_\ell \text{ for each } 0 \leq j, l \leq q\} \\ &\leq s_\ell^{(q-1)\beta} n^q; \end{aligned}$$

in this scale, P_ℓ is of lesser order than n^q, which means that (at least in this scale) the orbit of a typical point spreads fast (leading to a behavior which is similar to a hyperbolic system).

In summary, the orbit of a point $x \in Z$ has a very complex structure, being "tight" for some spatial scale, and spreading rapidly throughout the space for another scale.

Combining Corollary 2.1 and Theorem 2.1 with Proposition I.2, one gets the following result.

Corollary 2.3. *Let (X, T) be the full-shift system, $X = \prod_{-\infty}^{\infty} M$, where M is a perfect and compact metric space. Let X be endowed with the metric* (2.1). *Then,*

$$HP = \{\mu \in \mathcal{M}_e(T) \mid \dim_H^+(\mu) = 0 \textit{ and } \dim_P^-(\mu) = \infty\}$$

is a residual subset of $\mathcal{M}(T)$.

Again, one may compare Corollary 2.3 with Theorem 1.1(III-IV). Here, X is perfect, compact and endowed with the metric (2.1). There, X is perfect, Polish, and endowed with any metric such that T and T^{-1} are both Lipschitz.

By Corollary 2.3, each $\mu \in HP \cap C_X(T)$ is supported on a set $Z \subset X$ with $\dim_H(Z) = 0$ and $\dim_P(Z) = \infty$. Thus, Z is a dense and totally disconnected subset of X (suppose that Z is not dense; then, there exist $x \in X$ and $\varepsilon > 0$ such that $B(x, \varepsilon) \cap Z = \emptyset$. This results in $1 = \mu(Z) + \mu(B(x, \varepsilon))$, which is absurd since $\mu(B(x, \varepsilon)) > 0$).

Finally, we may also say something about the typical lower and upper entropy dimensions of an invariant measure. Combining Theorems 2.1 and 2.2 with Proposition I.2, the following result holds.

Corollary 2.4. *Each of the sets*

$$ED_- = \{\mu \in \mathcal{M}(T) \mid D_\mu^-(1) = 0\},$$

$$ED_+ = \{\mu \in \mathcal{M}(T) \mid D_\mu^+(1) = +\infty\}$$

is residual in $\mathcal{M}(T)$.

2.1 G_δ sets

In this section, X is always a compact metric space.

2.1.1 G_δ sets for $s \in (0,1)$

Let $\mu \in \mathcal{M}$, let $s \in (0,1)$ and let $\mathcal{G} = \{B(x_j, \varepsilon)\}$ be some countable covering of X by balls of radius $\varepsilon > 0$. Let $\tilde{\mathcal{G}} = \{B(x_i, \varepsilon)\} \subset \mathcal{G}$ be a sub-covering of X that also covers $\operatorname{supp}(\mu)$.

For each $x \in B(x_i, \varepsilon)$, one has $B(x_i, \varepsilon) \subset B(x, 2\varepsilon)$, from which follows that, for each $x \in B(x_i, \varepsilon) \cap \operatorname{supp}(\mu)$, $\mu(B(x_i, \varepsilon))^{s-1} \geq \mu(B(x, 2\varepsilon))^{s-1}$; hence,

$$\begin{aligned} I_\mu(s, 2\varepsilon) &= \int_{\operatorname{supp}(\mu)} \mu(B(x, 2\varepsilon))^{s-1} d\mu(x) \leq \sum_{x_i \in \tilde{\mathcal{G}}} \int_{B(x_i,\varepsilon)\cap \operatorname{supp}(\mu)} \mu(B(x, 2\varepsilon))^{s-1} d\mu(x) \\ &\leq \sum_{x_i \in \tilde{\mathcal{G}}} \int_{B(x_i,\varepsilon)\cap \operatorname{supp}(\mu)} \mu(B(x_i, \varepsilon))^{s-1} d\mu(x) = \sum_{x_i \in \tilde{\mathcal{G}}} \mu(B(x_i, \varepsilon))^s \\ &\leq \sum_{x_j \in \mathcal{G}} \mu(B(x_j, \varepsilon))^s \qquad (2.2) \end{aligned}$$

(by $x \in \mathcal{G}$ one means that $B(x, \varepsilon) \in \mathcal{G}$; we will use this notation throughout the text).

Naturally, since X is a compact metric space, one can assume, without loss of generality, that $\mathcal{G}$ is always a finite covering of X.

Definition 2.1. Let $\mu \in \mathcal{M}$. One defines, for each $s \in (0,1)$ and each $\varepsilon > 0$,

$$S_\mu(s,\varepsilon) = \inf_{\mathcal{G}} \sum_{x_j \in \mathcal{G}} \mu(B(x_j,\varepsilon))^s,$$

where the infimum is taken over all finite coverings, $\mathcal{G}$, of X by balls of radius ε (as above).

Remark 2.1. One must compare Definition 2.1 with Definition (8.6) in [42].

Definition 2.2. Let $\mu \in \mathcal{M}$. One defines, for each $s \in (0,1)$ and each $\varepsilon > 0$,

$$W_\mu(s,\varepsilon) = \inf_{\mathcal{G}} \sum_{x_j \in \mathcal{G}} f_\varepsilon(\mu, x_j)^s,$$

where the infimum is taken over all finite coverings, $\mathcal{G}$, of X by balls of radius ε, and $f_\varepsilon(\mu, x_j)$ is defined in the statement of Lemma 1.2.

Proposition 2.1. *Let $s \in (0,1)$ and let $\mu \in \mathcal{M}$. Then,*

$$d_\mu^-(s) := \liminf_{\varepsilon \to 0} \frac{\log W_\mu(s,\varepsilon)}{(s-1)\log\varepsilon} = \liminf_{\varepsilon \to 0} \frac{\log S_\mu(s,\varepsilon)}{(s-1)\log\varepsilon}.$$

Moreover, $D_\mu^-(s) \le d_\mu^-(s)$.

Proof. Let $\varepsilon > 0$. Then, one has

$$I_\mu(s,\varepsilon) \le S_\mu(s,\varepsilon/2) \le W_\mu(s,\varepsilon/2) \le S_\mu(s,\varepsilon),$$

from which the results follow. The first inequality above comes from (2.2). The remaining inequalities come from $\mu(B(x,\varepsilon/2))^s \le f_{\varepsilon/2}(\mu,x)^s \le \mu(B(x,\varepsilon))^s$, valid for each $x \in X$. □

Remark 2.2. One may compare Proposition 2.1 with Theorem 8.4 (1) in [42].

Proposition 2.2. *Let $\varepsilon > 0$, let $s \in (0,1)$, and let $\mathcal{G} = \{B(x_l,\varepsilon)\}_{l=1}^L$ be a finite covering of X by open balls of radius ε. Then, the function*

$$H_{\mathcal{G}} : \mathcal{M} \longrightarrow \mathbb{R}^+, \qquad H_{\mathcal{G}}(\mu) = \sum_{l=1}^{L} f_\varepsilon(\mu, x_l)^s,$$

is continuous in the weak topology.

Proof. Let (μ_n) be a sequence in $\mathcal{M}$ such that $\mu_n \to \mu$. Since, for each $l = 1, \ldots, L$, the mapping $\mathcal{M} \ni \mu \mapsto f_\varepsilon(\mu, x_l) \in \mathbb{R}_+$ is continuous (by Lemma 1.2), it follows that $H_{\mathcal{G}}(\mu) = \sum_{l \in L} f_\varepsilon(\mu, x_l)^s$ is also continuous, being a finite sum of continuous functions. □

Proposition 2.3. *Let $s \in (0,1)$. Then, $D_-^* = \{\mu \in \mathcal{M} \mid \ d_s^-(\mu) = 0\}$ is a G_δ subset of $\mathcal{M}$.*

Proof. Let $\mu \in \mathcal{M}$ and let $\varepsilon > 0$. Define $h : \mathcal{M} \to (0, +\infty)$ by the law $h(\mu) = W_\mu(s, \varepsilon) = \inf_{\mathcal{G}} \sum_{x_j \in \mathcal{G}} f_\varepsilon(\mu, x_j)^s$ (where the infimum is taken over all finite coverings, $\mathcal{G}$, of X by open balls of radius ε) and $g_\varepsilon : (0, +\infty) \to \mathbb{R}$ by the law $g_\varepsilon(r) = \frac{\log(r)}{(s-1)\log \varepsilon}$. Note that, for each $k \in \mathbb{N}$, $g_\varepsilon^{-1}((-\infty, 1/k)) = (0, a_k)$, where $a_k = g_\varepsilon^{-1}(1/k)$.

It follows from Proposition 2.2 that h is upper semicontinuous, and thus, for each $k \in \mathbb{N}$, $(g_\varepsilon \circ h)^{-1}((-\infty, 1/k)) = h^{-1}\left(g_\varepsilon^{-1}((-\infty, 1/k))\right) = h^{-1}((0, a_k))$ is open in $\mathcal{M}$. Since

$$\begin{aligned} D_-^* &= \left\{\mu \in \mathcal{M} \mid \liminf_{\varepsilon \to 0} \frac{\log W_\mu(s, \varepsilon)}{(s-1)\log \varepsilon} = 0\right\} \\ &= \bigcap_{k \in \mathbb{N}} \bigcap_{l \in \mathbb{N}} \bigcup_{t > l} \left\{\mu \in \mathcal{M} \mid \frac{\log W_\mu(s, 1/t)}{(s-1)\log 1/t} < \frac{1}{k}\right\} \\ &= \bigcap_{k \in \mathbb{N}} \bigcap_{l \in \mathbb{N}} \bigcup_{t > l} (g_{1/t} \circ h)^{-1}((-\infty, 1/k)), \end{aligned}$$

the result follows. □

2.1.2 G_δ sets for $q > 1$

Lemma 2.1. *Let, for each $q > 1$ and each $\varepsilon > 0$, $\mathcal{M} \ni \mu \mapsto J_\mu(q, \varepsilon) \in [0,1]$ be defined by the law*

$$J_\mu(q, \varepsilon) = \int f_\varepsilon(\mu, x)^{q-1} d\mu(x).$$

Then,

$$D_\mu^\pm(q) = \limsup_{\varepsilon \to 0}(\inf) \frac{\log J_\mu(q, \varepsilon)}{\log(\varepsilon)},$$

where $f_\varepsilon(\mu, x) = \int f_x^\varepsilon(y)d\mu(y)$ *is defined in the statement of Lemma 1.2. Moreover, the mapping* $\mathcal{M} \ni \mu \mapsto J_\mu(q,\varepsilon) \in [0,1]$ *is continuous.*

Proof. The proof is divided into the following steps.

Step 1. Note that, for each $\varepsilon \in (0,1)$, $I_\mu(q,\varepsilon) \le J_\mu(q,\varepsilon) \le I_\mu(q,2\varepsilon)$. Then,

$$\frac{\log I_\mu(q,\varepsilon)}{\log(\varepsilon)} \ge \frac{\log J_\mu(q,\varepsilon)}{\log(\varepsilon)} \ge \frac{\log I_\mu(q,2\varepsilon)}{\log(\varepsilon)},$$

from which follows that

$$D_\mu^\pm(q) = \lim_{\varepsilon\to 0}\sup(\inf)\frac{\log I_\mu(q,\varepsilon)}{(q-1)\log(\varepsilon)} = \lim_{\varepsilon\to 0}\sup(\inf)\frac{\log J_\mu(q,\varepsilon)}{(q-1)\log(\varepsilon)}.$$

Step 2. We prove that, for each $\varepsilon > 0$, the mapping $\mathcal{M} \ni \mu \mapsto J_\mu(q,\varepsilon) \in [0,1]$ is continuous. Let (μ_n) and (ν_m) be sequences in $\mathcal{M}$ such that $\mu_n \to \mu$ and $\nu_m \to \nu$. Set $J_{\mu,\nu}(q,\varepsilon) := \int \left(\int f_x^\varepsilon(y)d\mu(y)\right)^{q-1} d\nu(x)$. We shall prove that

$$\lim_{n,m\to\infty} J_{\mu_n,\nu_m}(q,\varepsilon) = \lim_{n,m\to\infty}\int\left(\int f_x^\varepsilon(y)d\mu_n(y)\right)^{q-1} d\nu_m(x) = J_{\mu,\nu}(q,\varepsilon).$$

Firstly, we show that

$$\lim_{m\to\infty}\lim_{n\to\infty}\int f_\varepsilon(\mu_n,x)^{q-1}d\nu_m(x) = J_{\mu,\nu}(q,\varepsilon).$$

Since $f_x^\varepsilon(\cdot)$ is continuous and $\mu_n \to \mu$, it follows that $\lim_{n\to\infty} f_\varepsilon(\mu_n,x) = \int f_x^\varepsilon(y)d\mu(y)$.

Clearly, for each $x \in X$ and each $n \in \mathbb{N}$, $|f_\varepsilon(\mu_n,x)|^{q-1} \le 1$; thus, by the Dominated Convergence Theorem,

$$\lim_{m\to\infty}\lim_{n\to\infty}\int f_\varepsilon(\mu_n,x)^{q-1}d\nu_m(x) \;=\; \lim_{m\to\infty}\int f_\varepsilon(\mu,x)^{q-1}d\nu_m(x). \qquad (2.3)$$

Note that, for each $\mu \in \mathcal{M}$, the mapping $X \ni x \mapsto f_\varepsilon(\mu,x) \in \mathbb{R}_+$ is continuous. Indeed, let (x_l) be a sequence in X such that $x_l \to x$. Since $f_{x_l}^\varepsilon(\cdot)$ converges uniformly to $f_x^\varepsilon(\cdot)$, and for each $y \in X$ and each $l \in \mathbb{N}$, $|f_{x_l}^\varepsilon(y)| \le 1$, it follows again from the Dominated

Convergence Theorem that

$$\lim_{l\to\infty} f_\varepsilon(\mu, x_l) = \lim_{l\to\infty} \int f^\varepsilon_{x_l}(y)d\mu(y) = \int f^\varepsilon_x(y)d\mu(y) = f_\varepsilon(\mu, x).$$

Thus, one gets from (2.3) that

$$\lim_{m\to\infty} \int f_\varepsilon(\mu, x)^{q-1} d\nu_m(x) = \int f_\varepsilon(\mu, x)^{q-1} d\nu(x) = J_{\mu,\nu}(q, \varepsilon).$$

Now, we show that for each $n \in \mathbb{N}$, the function $\varphi_n : \mathbb{N} \to \mathbb{N}$ defined by the law $\varphi_n(m) := J_{\mu_n,\nu_m}(q, \varepsilon)$, converges uniformly on $m \in \mathbb{N}$ to $\varphi(m) := \lim_{n\to\infty} J_{\mu_n,\nu_m}(q, \varepsilon) = \int f_\varepsilon(\mu, x)^{q-1} d\nu_m(x)$.

Namely, let $\delta > 0$ and let $m \in \mathbb{N}$. Since $(\mathcal{M}, r_1)\times(X, \rho)$ is compact and, by Lemma 1.2, $f_\varepsilon(\,\cdot\,,\,\cdot\,) : \mathcal{M}\times X \to [0, 1]$ is continuous, $f_\varepsilon(\,\cdot\,,\,\cdot\,)$ is in fact uniformly continuous on $\mathcal{M}\times X$. Note also that the function $h : [0, 1] \to [0, 1]$, given by the law $h(x) = x^{q-1}$, is continuous on $[0, 1]$; then, $h \circ f : \mathcal{M} \times X \to [0, 1]$ is uniformly continuous. Hence, there exists an $\eta > 0$ such that, for each $(\tilde{\mu}, \tilde{x}) \in \mathcal{M} \times X$ and each $(\mu, x) \in B((\tilde{\mu}, \tilde{x}), \eta) := \{(\nu, y) \in \mathcal{M} \times X \mid d((\tilde{\mu}, \tilde{x}), (\nu, y)) < \eta\}$, $|f_\varepsilon(\mu, x)^{q-1} - f_\varepsilon(\tilde{\mu}, \tilde{x})^{q-1}| < \delta$ ($\mathcal{M} \times X$ is endowed with the product metric $d((\mu, x), (\nu, y)) = r_1(\mu, \nu) + \rho(x, y)$, whose induced topology is equivalent to the product topology in $\mathcal{M} \times X$).

Since $\mu_n \to \mu$, there exists an $N \in \mathbb{N}$ such that, for each $n > N$, $r_1(\mu_n, \mu) < \eta$. Thus, for each $x \in X$ and each $n > N$, $d((\mu_n, x), (\mu, x)) = r_1(\mu_n, \mu) + \rho(x, x) < \eta$, which results in $(\mu_n, x) \in B((\mu, x), \eta)$. Thus, by the uniform continuity of $h \circ f$, it follows that, for each $x \in X$ and each $n > N$, $\left|\left(\int f^\varepsilon_x(y)d\mu_n(y)\right)^{q-1} - \left(\int f^\varepsilon_x(y)d\mu(y)\right)^{q-1}\right| < \delta$. Then, for each

$n > N$ and each $m \in \mathbb{N}$,

$$\begin{aligned} |\varphi_n(m) - \varphi(m)| &= \left| \int f_\varepsilon(\mu_n, x)^{q-1} d\nu_m(x) - \int f_\varepsilon(\mu, x)^{q-1} d\nu_m(x) \right| \\ &\le \int \left| \left(\int f_x^\varepsilon(y) d\mu_n(y) \right)^{q-1} - \left(\int f_x^\varepsilon(y) d\mu(y) \right)^{q-1} \right| d\nu_m(x) \\ &< \int \delta \, d\nu_m(x) \\ &= \delta. \end{aligned}$$

This proves that $\varphi_n(m) \to \varphi(m)$ uniformly on $m \in \mathbb{N}$. It follows, therefore, from Theorem 2.15 in [23] that

$$\lim_{n,m\to\infty} J_{\mu_n,\nu_m}(q,\varepsilon) = \lim_{n,m\to\infty} \int \left(\int f_x^\varepsilon(y) d\mu_n(y) \right)^{q-1} d\nu_m(x) = J_{\mu,\nu}(q,\varepsilon).$$

Since $J_\mu(q,\varepsilon)$ is the restriction of $J_{\mu,\nu}(q,\varepsilon)$ to the diagonal set $D \subset \mathcal{M} \times \mathcal{M}$, one gets

$$\lim_{n\to\infty} J_{\mu_n,\mu_n}(q,\varepsilon) = \lim_{n\to\infty} J_{\mu_n}(q,\varepsilon) = J_\mu(q,\varepsilon).$$

This show that the mapping $\mathcal{M} \ni \mu \mapsto J_\mu(q,\varepsilon) \in [0,1]$ is continuous in the weak topology. $\square$

Proposition 2.4. *Let $\alpha > 0$ and $q > 1$. Then, each of the sets*

$$D_+ = \{\mu \in \mathcal{M} \mid D_\mu^+(q) \ge \alpha\}$$

$$D_- = \{\mu \in \mathcal{M} \mid D_\mu^-(q) = 0\}$$

is G_δ subset of $\mathcal{M}$.

Proof. We just prove the first statement, given that the proof of the second one is completely analogous. Let $\mu \in \mathcal{M}$. It follows from Lemma 1.2 that, for each $\varepsilon > 0$,

$$\liminf_{t\to\infty} t^{\alpha(q-1)} J_\mu(q, 1/t) = 0 \;\Rightarrow\; D_\mu^+(q) \ge \alpha \;\Rightarrow\; \liminf_{t\to\infty} t^{(\alpha+\varepsilon)(q-1)} J_\mu(q, 1/t) = 0,$$

which results in

$$\bigcap_{n>0}\bigcap_{k>0}\bigcup_{t>k}\left\{\mu\in\mathcal{M}\mid t^{\alpha(q-1)}J_\mu(q,1/t)<\frac{1}{n}\right\}\subseteq\left\{\mu\in\mathcal{M}\mid D^+_\mu(q)\geq\alpha\right\}$$
$$\subseteq\bigcap_{n>0}\bigcap_{k>0}\bigcup_{t>k}\left\{\mu\in\mathcal{M}\mid t^{(\alpha+\varepsilon)(q-1)}J_\mu(q,1/t)<\frac{1}{n}\right\}.$$

Replacing α by $\alpha-\varepsilon$ in the last paragraph and taking $\varepsilon=\frac{1}{l}$, one gets

$$\begin{aligned}\bigcap_{l>0}\bigcap_{n>0}\bigcap_{k>0}\bigcup_{t>k}\left\{\mu\in\mathcal{M}\mid t^{(\alpha+\frac{1}{l})(q-1)}J_\mu(q,1/t)<\frac{1}{n}\right\} &= \bigcap_{l>0}\left\{\mu\in\mathcal{M}\mid D^+_\mu(q)\geq\alpha-\frac{1}{l}\right\}\\ &= \left\{\mu\in\mathcal{M}\mid D^+_\mu(q)\geq\alpha\right\}.\end{aligned}$$

Now, one just needs to prove that, for each $k,l,n\in\mathbb{N}$ and each $t>k$,

$$\left\{\mu\in\mathcal{M}\mid t^{(\alpha+\frac{1}{l})(q-1)}J_\mu(q,1/t)<\frac{1}{n}\right\}=\left(t^{(\alpha+\frac{1}{l})(q-1)}J_{(.)}(q,1/t)\right)^{-1}([0,1/n))$$

is an open set in $\mathcal{M}$; this is a direct consequence of Lemma 1.2. □

2.2 Dense sets

Proposition 2.5. *Let (X,T) be a topological dynamical system, assume that $\mathcal{M}_p(T)$ is dense in $\mathcal{M}(T)$, and let $s\in(0,1)$. Then, $D^*_-=\{\mu\in\mathcal{M}(T)\mid d^-_s(\mu)=0\}$ is a dense subset of $\mathcal{M}(T)$.*

Proof. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a dense subset of $\mathcal{M}_p(T)$ (recall that $\mathcal{M}_p(T)$ is separable), and let $\mu\in\{\mu_n\}_{n\in\mathbb{N}}$ be a T-periodic measure associated with the T-periodic point $x\in X$, whose period is k_x. Set $\varepsilon_0=\min_{0\leq i\neq j\leq k_x-1}\{d(x_i,x_j)\mid x_l:=T^l(x),\ l=0,\ldots,k_x-1\}$, set $A=\{x,T(x),\cdots,T^{k_x-1}(x)\}$, and let $\varepsilon\in(0,\min\{1,\varepsilon_0\})$.

As X is a compact metric space and $C=X\setminus\bigcup_{z\in A}B(z,\varepsilon)$ is closed, C is also compact. Let $\mathcal{G}_1=\{B(y_n,\varepsilon)\}_{y_n\in C}$ be a finite covering of C, and set $\tilde{\mathcal{G}}=\mathcal{G}_1\cup\{B(z,\varepsilon)\}_{z\in A}$. By construction, each $z \in A$ belongs to only one element of $\tilde{\mathcal{G}}$ (namely, $B(z,\varepsilon)$), and for each $y_n \in \mathcal{G}_1$, $\mu(B(y_n,\varepsilon)) = 0$. Thus,

$$S_\mu(s,\varepsilon) = \inf_{\mathcal{G}} \sum_{z_j \in \mathcal{G}} \mu(B(z_j,\varepsilon))^s \leq \sum_{w \in \tilde{\mathcal{G}}} \mu(B(w,\varepsilon))^s = k_x^{1-s},$$

from which follows that

$$\frac{\log S_\nu(s,\varepsilon)}{(s-1)\log\varepsilon} \leq \frac{\log(k_x^{1-s})}{(s-1)\log\varepsilon}.$$

Letting, $\varepsilon \to 0$, one gets $d_s^-(\nu) = 0$. □

Remark 2.3. The fact that $\mathcal{M}_p(T)$ is dense in $\mathcal{M}(T)$ is true, in particular, for the full-shift over $X = \prod_{-\infty}^{\infty} M$, where M is a Polish space, as we have seen in Chapter I

From now on, we endow $X = M^{\mathbb{Z}}$ with the following metric (which corresponds to the choice $a_n := n^2$, $n \in \mathbb{N} \cup \{0\}$, in (2.1)):

$$r(x,y) = \sum_{|n| \geq 0} \min\left\{\frac{1}{n^2+1}, d(x_n,y_n)\right\}.$$

Remark 2.4. Although we use this metric in what follows, the results presented below are also valid for any sub-exponential metric as defined by (2.1). We have made this particular choice in order to simplify the exposition of the main arguments (see also Remark 2.5).

Next, we prove that $CD = \{\mu \in \mathcal{M}(T) \mid D_\mu^+(q) = +\infty\}$ is a dense subset of $\mathcal{M}(T)$. Our strategy involves a modified version of the energy function (5): for each $q > 1$, each $\varepsilon > 0$, each $n \in \mathbb{N}$ and each $\mu \in \mathcal{M}(T)$, set

$$I_\mu^n(q,\varepsilon) := \int \mu(B^n(x,\varepsilon))^{q-1} d\mu(x),$$

where $B^n(x,\varepsilon) := \cdots \times M \times \cdots \times M \times B_M(x_{-n},\varepsilon) \times \cdots \times B_M(x_n,\varepsilon) \times M \times \cdots \times M \times \cdots$, and $B_M(z,\varepsilon) := \{w \in M \mid d(w,z) < \varepsilon\}$.

Lemma 2.2. *Let $\varepsilon > 0$. Then, there exists an $n_0 \in \mathbb{N}$ such that, for each $x \in X$, $B(x,\varepsilon) \subseteq B^{n_0}(x,\varepsilon)$.*

Proof. Let $x \in X$ and let $y \in B(x,\varepsilon)$; then, for each $n \in \mathbb{Z}$, $\min\{\frac{1}{n^2+1}, d(x_n,y_n)\} < \varepsilon$. Set $n_0 := [(\frac{1}{\varepsilon}-1)^{1/2}]+1$. Since $\frac{1}{(n_0+1)^2+1} \le \varepsilon < \frac{1}{n_0^2+1}$, it follows that, for each $|n| < n_0$, $\min\{\frac{1}{n^2+1}, d(x_n,y_n)\} = d(x_n,y_n) < \varepsilon$. Therefore, $y \in B^{n_0}(x,\varepsilon)$. □

The following result is a direct consequence of Lemma 2.2.

Proposition 2.6. *Let $q > 1$. Then,*

$$D_\mu^+(q) = \limsup_{\varepsilon\to 0} \frac{\log I_\mu(q,\varepsilon)}{(q-1)\log\varepsilon} \ge \widetilde{D}_\mu^+(q) := \limsup_{\varepsilon\to 0} \frac{\log I_\mu^{n_0}(q,\varepsilon)}{(q-1)\log\varepsilon},$$

where $n_0 = n_0(\varepsilon)$ is given by Lemma 2.2.

In what follows, X is a perfect and compact metric space.

Lemma 2.3. *Let $\mu \in \mathcal{M}(T)$ and let U be an open basic (weak) neighborhood of μ. Then, there exist $m_0, n_0 \in \mathbb{N}$ such that for each $m \ge m_0$ and each $n \ge n_0$, $\mu_x \in U \cap \mathcal{M}(T)$, where $x = (x_i) \in X$ is a T-periodic point with period $s = mn$ and $x_i \ne x_j$ if $i \ne j$, $i,j = 1,\dots,s$.*

Proof. We present the proof in details for the reader's sake. For each $k \in \mathbb{N}$, let π_k denote the projection of X onto $\prod_{-k}^{k} M$ and let

$$C_k(X) = \{f \mid f \in C(X) \ \text{ and if } \ \pi_k(x) = \pi_k(y), \ \text{ then } \ f(x) - f(y) = 0\}.$$

In other words, $C_k(X)$ is the set of functions $f \in C(X)$ which depend only on the coordinates $(x_{-k},\dots,x_k)\cdot$. The functions that belong to $C_0 = \bigcup_{k\ge 1} C_k(X)$ are called finite-dimensional.

Consider an arbitrary basic open neighborhood of μ in $\mathcal{M}(T)$, that is, a set of the form

$$U = \left\{\nu \in \mathcal{M}(T) \mid f \in F \to \left|\int f d\mu - \int f d\nu\right| < \varepsilon\right\},$$

where $\varepsilon > 0$ and F is a finite subset of $C(X)$. Since C_0 is dense in $C(X)$, one may (and shall) assume that $F \subset C_k(X)$, for some k.

Let $\{Q_1, \ldots, Q_N\}$ be a partition of Q into Borel sets of positive μ-measure on each of which the oscillation of $f^*(x) := \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(T^i x)$, for each $f \in F$, is less than $\varepsilon/2$ (here, Q is the set of quasi-regular points, that is, those points for which $f^*(x)$ is defined for every $f \in C(X)$). Choose points $x^j = \left\{x_i^j\right\} \in Q_j$ $(j = 1, \ldots, N)$. Then, for each $f \in F$,

$$\left|\int_Q f^* d\mu - \sum_{j=1}^{N} f^*\left(x^j\right)\mu\left(Q_j\right)\right| < \frac{\varepsilon}{2}.$$

Now, by a theorem of Kryloff and Bogoliouboff (see [37], p. 118), $\mu(Q) = 1$, and by the Ergodic Theorem, it follows that $\int_Q f^* d\mu = \int_Q f d\mu$.

Set, for each $n \in \mathbb{N}$, $f_n(x) = \frac{1}{n}\sum_{i=1}^{n} f(T^i x)$. Hence, there exists $n_0 \in \mathbb{N}$ such that, for each $n \geq n_0$ and each $f \in F$,

$$\left|\int f d\mu - \sum_{j=1}^{N} f_n\left(x^j\right)\mu\left(Q_j\right)\right| < \frac{\varepsilon}{2}$$

and

$$\frac{(2k+1)(2L)}{n} < \frac{\varepsilon}{2},$$

where $L = \max\{\|f\| \mid f \in F\}$. Fix $n \geq n_0$ and note that there exists $m_0 \in \mathbb{N}$ such that, for each $m \geq m_0$ and each $f \in F$, one can approximate the numbers $\mu\left(Q_j\right)$ by positive rational numbers m_j/m such that

$$\left|\int f d\mu - \sum_{j=1}^{N} f_n\left(x^j\right)\frac{m_j}{m}\right| < \frac{\varepsilon}{2} \tag{2.4}$$

and

$$\sum_{j=1}^{N} m_j = m.$$

For each $j = 1, \ldots, N$, denote by B_j the n-block

$$B_j = \left[x_1^j, \ldots, x_n^j\right],$$

and form the mn-block

$$B = \underbrace{B_1 \dots B_1}_{m_1} \underbrace{B_2 \dots B_2}_{m_2} \ \cdots \ \underbrace{B_N \dots B_N}_{m_N}.$$

Let x be the point of X with T-period mn such that

$$[x_1, \dots, x_{mn}] = B.$$

Thus, for each $f \in C_k(X)$, and therefore, for each $f \in F$, one has

$$f\left(T^i x\right) = f\left(T^i x^1\right) \quad \text{for} \quad k+1 \leq i \leq n-k.$$

By a simple procedure, one gets

$$\left| f_{mn}(x) - \sum_{j=1}^{N} f_n\left(x^j\right) \frac{m_j}{m} \right| \leq \frac{(2k+1)}{n}(2L) < \frac{\varepsilon}{2}. \tag{2.5}$$

Now, since x has T-period mn, it follows that

$$\int f d\mu_x = f_{mn}(x). \tag{2.6}$$

Finally, by combining (2.4), (2.5) and (2.6), it follows that, for each $f \in F$,

$$\left| \int f d\mu_x - \int f d\mu \right| < \varepsilon.$$

Therefore, $\mu_x \in U \cap \mathcal{M}(T)$, where x is a T-periodic point of period $s = mn \geq m_0 n_0 =: s_0$. In order to complete the proof, just note that since each point of M is a limit point and since each $f \in F$ is continuous, one can choose x such that $x_i \neq x_j$ if $i \neq j$, $i, j = 1, \dots, s$, keeping the estimates as before. $\square$

Proposition 2.7. *Let $\mu \in \mathcal{M}(T)$ and let $q > 1$. Then, each (weak) neighborhood, V, of μ contains $\rho \in \mathcal{M}(T)$ such that $D_\rho^+(q) = +\infty$.*

Proof. Let $\delta > 0$ and set

$$V = V_\mu(f_1, \cdots, f_d; \delta) = \left\{\sigma \in \mathcal{M} \mid \left| \int f_j d\mu - \int f_j d\sigma \right| < \delta,\ j = 1, \ldots, d \right\},$$

where each $f_j \in C(M^{\mathbb{Z}})$ (this is the set of continuous real valued functions on $M^{\mathbb{Z}}$, endowed with the supremum norm). One can further assume that there exists an N such that, for each $j = 1, \ldots, d$, one has $f_j(x) = f_j(y)$ if, for each $|i| \le N$, $x_i = y_i$. Note that, since M is compact, functions of this type form a dense set in $C(M^{\mathbb{Z}})$.

Let $L = \sup\{|f_j(x)| \mid x \in M^{\mathbb{Z}},\ j = 1, \ldots, d\}$, let $\kappa > 0$ be such that

$$\begin{aligned} \kappa &< (8L)^{-1}\, 2^{-(2N+1)}\, \delta, \\ 1 - (1-\kappa)^{2N} &< (8L)^{-1}\delta, \end{aligned} \tag{2.7}$$

and set $S = 1 + \left(\frac{\kappa^q}{1-(1-\kappa)^q}\right)^{1/(q-1)}$. It follows from Lemma 2.3 that there exists a T-periodic point $w = (w_i) \in M^{\mathbb{Z}}$, with period $s = mn$, where $m \ge \max\{m_0, S\}$ and $n \ge \max\{n_0, S\}$, such that for each $i \ne j$, $i, j = 1, \ldots, s$, $w_i \ne w_j$ and $\mu_w \in V_\mu(f_1, \cdots, f_d; \delta/2)$.

Following the proof of Lemma 7 in [51], one defines, for each fixed $s \ge s_0$, a Markov chain ρ whose states are $w_1, \cdots, w_s$, whose initial probabilities are given by the s-tuple $(1/s, \cdots, 1/s)$, and whose transition probabilities are given by the $s \times s$-matrix p_{ij}, where

$$\begin{aligned} p_{s\,1} &= 1 - \kappa, \\ p_{i\,i+1} &= 1 - \kappa \quad \text{for } i = 1, \ldots, s-1, \\ p_{i\,j} &= \frac{\kappa}{s-1} \quad \text{otherwise.} \end{aligned}$$

One can show (see the proof of Lemma 7 in [51]) that $\rho \in V_{\mu_w}(f_1, \cdots, f_d; \delta/2)$, from which follows that $\rho \in V_\mu(f_1, \cdots, f_d; \delta)$.

Now, by Proposition 2.6, one just needs to prove that $\widetilde{D}^+_\rho(q) = \infty$. Let $\varepsilon \in (0, \min\{1, \varepsilon_0\})$, with $\varepsilon_0 := \min\{|w_i - w_l| : i, l = 1, \ldots, s\}$, and set $n = n_0(\varepsilon)$.

Set $C^n = [-n; a_{i_{-n}}, \ldots, a_{i_n}] = \{(y_i)_{i\in\mathbb{Z}} \in X \mid y_{-n} = a_{i_{-n}}, \ldots, y_n = a_{i_n}\}$, with

$a_{i_{-n}}, \ldots, a_{i_n} \in \{w_1, \ldots, w_s\}$. For each $x \in C^n$, it is clear from the choice of ε that $C^n \subset B^n(x, \varepsilon) = \{(y_i)_{i \in \mathbb{Z}} \in X \mid y_i \in B(x_i, \varepsilon),\ i = -n, \ldots, n\}$ and that $\rho(B^n(x, \varepsilon)) = \rho(C^n)$.

Note that, as in Lemma 7 in [51], there are s^{2n+1} sets of the form $C^n = [-n; a_{i_{-n}}, \ldots, a_{i_n}]$ (which we will refer as the (n-th level) cylinders) that can be split into two groups, say P and Q. Then P consists of those s sets which contain an element of the orbit of w. The second group, Q, splits into the groups $Q_1, \cdots, Q_{2n}$, where Q_p is the group of those $s\binom{2n}{p}(s-1)^p$ (n-th level) cylinders for which there are exactly p places $i = -n, \ldots, n$ where a_{i+1} is not the *natural follower* of a_i, in the sense that if $a_i = w_l$ and $a_{i+1} = w_m$, then $m \neq l + 1 (\operatorname{mod} s)$. For each $j = 1, \ldots, s^{2n+1}$, denote by C_j^n these (n-th level) cylinders.

Thus, since $I_\rho^n(q, \varepsilon)$ depends only on the values taken by $\rho(B^n(x, \varepsilon))$ when x ranges over the s^{2n+1} (n-th level) cylinders described above, one has

$$\begin{aligned}
\int \rho(B^n(x,\varepsilon))^{q-1} d\rho(x) &= \sum_{j=1}^{s^{2n+1}} \int_{C_j^n} \rho(B^n(x,\varepsilon))^{q-1} d\rho(x) + \int_{X \setminus \bigcup_{j=1}^{s^{2n+1}} C_j^n} \rho(B^n(x,\varepsilon))^{q-1} d\rho(x) \\
&= \sum_{j=1}^{s^{2n+1}} \int_{C_j^n} \rho(C_j^n)^{q-1} d\rho(x) = \sum_{j=1}^{s^{2n+1}} \rho(C_j^n)^{q-1} \rho(C_j^n) \\
&= \sum_{j=1}^{s^{2n+1}} \rho(C_j^n)^q = \sum_{C^n \in P} \rho(C^n)^q + \sum_{p=1}^{2n} \sum_{C^n \in Q_p} \rho(C^n)^q \\
&= s\left(\frac{1}{s}(1-\kappa)^{2n}\right)^q + \sum_{p=1}^{2n} \sum_{C^n \in Q_p} \frac{1}{s^q} p_{a_{-n}, a_{-n+1}}^q \cdots p_{a_{n-1}, a_n}^q, \qquad (2.8)
\end{aligned}$$

where we have used, in the second inequality, that for each $x \in C^n$ and each $0 < \varepsilon < \varepsilon_0$, $\rho(B^n(x, \varepsilon)) = \rho(C^n)$, as previously discussed.

Now,

$$\sum_{C^n \in Q_p} \frac{1}{s^q} p_{a_{-n}, a_{-n+1}}^q \cdots p_{a_{n-1}, a_n}^q = s\binom{2n}{p}(s-1)^p \cdot \frac{1}{s^q}\left(\frac{\kappa}{s-1}\right)^{pq}(1-\kappa)^{(2n-p)q},$$

and therefore,

$$\sum_{p=1}^{2n}\sum_{C^n\in Q_p}\frac{1}{s^q}p^q_{a_{-n},a_{-n+1}}\cdots p^q_{a_{n-1},a_n} = s^{1-q}\sum_{p=1}^{2n}\binom{2n}{p}(s-1)^p\left(\frac{\kappa^q}{(s-1)^q}\right)^p((1-\kappa)^q)^{(2n-p)}$$
$$= s^{1-q}\left(\left((s-1)^{1-q}\kappa^q+(1-\kappa)^q\right)^{2n}-(1-\kappa)^{2nq}\right) \quad (2.9)$$

Thus, combining (2.8) with (2.9), one gets

$$\int \rho(B^n(x,\varepsilon))^{q-1}d\rho(x) = s^{1-q}\left[(1-\kappa)^{2nq}+\left((s-1)^{1-q}\kappa^q+(1-\kappa)^q\right)^{2n}-(1-\kappa)^{2nq}\right]$$
$$= s^{1-q}\left((s-1)^{1-q}\kappa^q+(1-\kappa)^q\right)^{2n}.$$

Recall that, by Lemma 2.2, one has $n\geq(\frac{1}{\varepsilon}-1)^{1/2}-1$. Note also that $\log\left((s-1)^{1-q}\kappa^q+(1-\kappa)^q\right)<0$ by the definition of S. Thus,

$$\log\int \rho(B^n(x,\varepsilon))^{q-1}d\rho(x) = \log\left(s^{1-q}\left((s-1)^{1-q}\kappa^q+(1-\kappa)^q\right)^{2n}\right)$$
$$= (1-q)\log s+2n\log\left((s-1)^{1-q}\kappa^q+(1-\kappa)^q\right)$$
$$\leq (1-q)\log s+(2(1/\varepsilon-1)^{1/2}-2)\log\left((s-1)^{1-q}\kappa^q+(1-\kappa)^q\right),$$

from which follows that

$$\frac{\log\int \rho(B^n(x,\varepsilon))^{q-1}d\rho(x)}{(q-1)\log\varepsilon} \geq \frac{(1-q)\log s}{(q-1)\log\varepsilon}-\frac{2\log\left((s-1)^{1-q}\kappa^q+(1-\kappa)^q\right)}{(q-1)\log\varepsilon}+$$
$$\frac{2\log\left((s-1)^{1-q}\kappa^q+(1-\kappa)^q\right)}{(q-1)}\frac{(1/\varepsilon-1)^{1/2}}{\log\varepsilon}. \quad (2.10)$$

Letting $\varepsilon\to 0$, one gets $\widetilde{D}^+_\rho(q)=+\infty$. □

Remark 2.5. It is clear from inequality (2.10) that the metric r for which the previous result is valid must necessarily be sub-exponential, since in this case, $\lim_{\varepsilon\to 0}\frac{h(1/\varepsilon)}{|\log\varepsilon|}=+\infty$, where h is the inverse of the (invertible) function $f:[0,\infty)\to(0,\infty)$, defined in such a way that, for each $n\in\mathbb{N}\cup\{0\}$, $f(n):=a_n$ (see the discussion immediately after (2.1)).

Moreover, if one considers the exponential metric $r(x,y)=\sum_{|k|\geq 0}\min\{2^{-|k|},d(x_k,y_k)\}$, or even $r(x,y)=\sum_{|k|\geq 0}2^{-|k|}\frac{d(x_k,y_k)}{1+d(x_k,y_k)}$ (naturally, one can replace $a_n=2^{-|n|}$ by $a_n=c^{-\alpha|n|}$, with $c > 1$ and $\alpha > 0$), then for each $q > 1$,

$$D_\rho^+(q) \leq \frac{2|\log((s-1)^{1-q}\kappa^q + (1-\kappa)^q)|}{(q-1)\log 2}, \tag{2.11}$$

with ρ, κ and s defined as in the proof of Proposition 2.7.

Namely, if $n_0 \in \mathbb{N}$ is such that $2^{-n_0} < \varepsilon \leq 2^{-n_0+1}$, then it is easy to see that for each $n \geq n_0$ and each $x \in X$, $C^n(x) \subset B(x;\varepsilon)$; thus, as in equation (2.8),

$$\begin{aligned} \int \rho(B(x,\varepsilon))^{q-1}d\rho(x) &= \sum_{j=1}^{s^{2n+1}} \int_{C_j^n} \rho(B(x,\varepsilon))^{q-1}d\rho(x) + \int_{X\setminus \bigcup_{j=1}^{s^{2n+1}} C_j^n} \rho(B(x,\varepsilon))^{q-1}d\rho(x) \\ &\geq \sum_{j=1}^{s^{2n+1}} \int_{C_j^n} \rho(C_j^n)^{q-1}d\rho(x) = s^{1-q}\left((s-1)^{1-q}\kappa^q + (1-\kappa)^q\right)^{2n}, \end{aligned}$$

from which follows that (for $\varepsilon \in (0, \min\{1, \varepsilon_0\})$ and $n = n_0$)

$$\frac{\log \int \rho(B(x,\varepsilon))^{q-1}d\rho(x)}{(q-1)\log\varepsilon} \leq \frac{\log s}{|\log\varepsilon|} + \frac{2(|\log\varepsilon| + \log 2)|\log\left((s-1)^{1-q}\kappa^q + (1-\kappa)^q\right)|}{(q-1)(\log 2)|\log\varepsilon|}$$

Letting $\varepsilon \to 0$, one gets (2.11). In particular, given $\eta > 0$, there exists a dense set of the Markov shifts ρ such that $D_\rho^+(q) < \eta$; namely, just choose κ small enough and s large enough so that $|\log((s-1)^{1-q}\kappa^q + (1-\kappa)^q)| < (\eta(q-1)\log 2)/2$, and we are done.

2.3 Proof of the Theorems 2.1 and 2.2

***Proof* (Theorem 2.1)**. Since, by Proposition 2.1,

$$D_-^* = \{\mu \in \mathcal{M}(T) \mid \ d_q^-(\mu) = 0\} \subset FD = \{\mu \in \mathcal{M}(T) \mid \ D_q^-(\mu) = 0\},$$

the result follows from Propositions 2.3 and 2.5. □

***Proof* (Theorem 2.2).** The result is a direct consequence of Propositions 2.4, 2.6 and 2.7. □

Remark 2.6. It follows from Remark 2.5 that if Theorem 2.2 is true for the product space X endowed with an exponential metric, then the proof will follow from a different argument than the one presented in the proof of Proposition 2.7.

CHAPTER III

DIMENSION OF INVARIANT MEASURES OF EXPANSIVE HOMEOMORPHISMS AND AXIOM A SYSTEMS

Again, as in the previous chapters, we begin making some comments and observations about the results obtained here, as well as some of their dynamical and topological consequences. The proofs of central results (Theorems 3.1, 3.4, Proposition 3.1 and Corollary 3.6), are presented in Sections 3.1 and 3.2.

Our first result in this chapter present, under some hypotheses, lower and upper bounds for the generalized fractal dimensions of any Borel probability measure defined on a compact metric space.

Theorem 3.1. *Let X be a compact metric space, let μ be a probability Borel measure on X and suppose that there exist constants $\alpha < \beta \in (0, \infty)$ such that, for each $x \in \mathrm{supp}(\mu)$, $\alpha \leq \underline{d}_\mu(x) \leq \overline{d}_\mu(x) \leq \beta$. Then, for each $s < 1$ and each $q > 1$, one has*

$$\alpha \leq D_\mu^-(q) \leq D_\mu^-(1) \leq D_\mu^+(1) \leq D_\mu^+(s) \leq \beta.$$

It follows from Lemma 3.1 that f-homogeneous measures (see Definition I.22) satisfy the hypotheses of Theorem 3.1. In particular, the next result, which is already known in the literature (see Theorem 2.5 in [54] and [42]), is an extension of Young's formula ([59], Theorem 3.1) to the generalized fractal dimensions of the Bowen-Margulis measure (see Introduction) associated to a $C^{1+\alpha}$-Axiom A system over a two-dimensional compact Riemannian manifold.

Proposition 3.1. *Let $T : X \to X$ be a $C^{1+\alpha}$-Axiom A system ($\alpha > 0$) over a two-dimensional compact Riemannian manifold M (as in Remark I.2). Suppose that μ is its Bowen-Margulis measure and let $\lambda_1(\mu) \geq \lambda_2(\mu)$ be its Lyapounov exponents. Then, for each $q \in \mathbb{R}$,*

$$D_\mu^+(q) = D_\mu^-(q) = h_\mu(T) \left[\frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right]. \tag{3.1}$$

Definition 3.1. Let X be a metrizable space and let $f : X \to X$ be a homeomorphism. One says that d is a hyperbolic metric for X if there exist numbers $k > 1$ and $\varepsilon > 0$ such that, for each $x, y \in X$,

$$\max\{d(f(x), f(y)), d(f^{-1}(x), f^{-1}(y))\} \geq \min\{k\, d(x, y), \varepsilon\}. \tag{3.2}$$

Moreover, both f and f^{-1} are Lipschitz for d.

The following results show that if X is a metrizable compact space, then a homeomorphism $f : X \to X$ is expansive if and only if X admits a hyperbolic metric (see Definition I.21).

Theorem 3.2 (Theorem 5.1 in [18])**.** *If $f : X \to X$ is an expansive homeomorphism of the compact metric space X, then there exist a hyperbolic metric for X which is compatible with its topology.*

Theorem 3.3 (Theorem 5.3 in [18])**.** *Let $f : X \to X$ be a homeomorphism of the metrizable compact space X. Suppose that there exists a metric d on X defining its topology and numbers $k > 1, \varepsilon > 0$ such that, for each $x, y \in X$,*

$$\max \left(d(f(x), f(y)), d\left(f^{-1}(x), f^{-1}(y)\right) \right) \geqq \min(kd(x, y), \varepsilon).$$

Then, f is expansive and

$$(\dim_H)_d(X) \leqq \overline{C}_d(X) \leqq 2 \frac{h(f)}{\log k},$$

where $\overline{C}_d(X)$ and $(\dim_H)_d(X)$ are, respectively, the upper capacity and the Hausdorff *dimension of X with respect to d; $h(f)$ stands for the topological entropy of f (see* (C.1)*). In particular, $\overline{C}_d(X)$ and $(\dim_H)_d(X)$ are finite.*

The next results present, for an expansive homeomorphism over a compact metric space, some estimates for the generalized fractal dimensions of its invariant measures in terms of the metric entropy (see Appendix C for the definitions).

Theorem 3.4. *Let $f : X \to X$ be an expansive homeomorphism over a compact metric space X, and let d be the respective hyperbolic metric. Then, for each invariant measure $\mu \in \mathcal{M}(f)$ and each $q > 1$ one has $D^+_\mu(q) \leq h_\mu(f) \log k$.*

Theorem 3.5 (Theorem 6 in [50])**.** *Let $f : M \to M$ be an Axiom A diffeomorphism. Then,*

$$\mathcal{M}_z = \{\mu \in \mathcal{M}(f) \mid h_\mu(f) = 0\}$$

is a residual subset of $\mathcal{M}(f)$.

One may combine Theorem 3.4 with Theorem 3.5 in order to obtain the following result.

Theorem 3.6. *Let $T : X \to X$ be a C^1-Axiom A, and let $q \geq 1$. Then, the set*

$$CD_0 = \{\mu \in \mathcal{M}(T) \mid D^+_\mu(q) = 0\}$$

is generic in $\mathcal{M}(T)$.

Theorem 3.6 may be combined with Proposition I.2 in order to produce the following result. Let $q \geq 1$; if $\mu \in CD_0$, then there exists a Borel set $Z \subset X$, $\mu(Z) = 1$, such that for each $x \in Z$, one has $\alpha_q(x) = D_\mu(q) = 0$.

This means that if $x \in Z$, since $\alpha_q(x) = 0$, it follows that given $0 < \alpha \ll 1$ and $R > 0$, there exist $\delta > 0$ such that if $0 < |\varepsilon| < \beta = \min\{\delta, R\}$, then there exists $N = N(x, \alpha, \beta) \in \mathbb{N}$ such that, for each $n > N$, one has $C_q(x, n, \varepsilon) \geq \varepsilon^{(q-1)\alpha}$. Thus, we have that

$$\begin{aligned}\gamma &= \text{card}\,\{(i_1\cdots i_q)\in\{0,1,\cdots,n\}^q \mid d(T^{i_j}(x),T^{i_l}(x))\le\varepsilon \ \text{ for each } \ 0\le j,l\le q\}\\ &\ge \varepsilon^{(q-1)\alpha}\,n^q;\end{aligned}$$

thus, the quantity γ is of order n^q for each n large enough. This means, as we have seen, that the orbit of a μ-typical point is similar to a periodic orbit.

The next result in another direct consequence of Theorem 3.4.

Corollary 3.1. *Let X be a compact metric space, let $f: X\to X$ be a homeomorphism and let $q\ge 1$. If there exist a hyperbolic metric d compatible with the topology of X and $\mu\in\mathcal{M}(f)$ such that $D^+_\mu(q)>0$, then $h(f)\ge h_\mu(f)>0$.*

3.1 Generalized fractal dimensions for f-homogeneous measures

In this section we prove, for an f-homogeneous measure, some estimates on the generalized fractal dimensions of such measure in terms of its Lyapunov exponents (for a hyperbolic measure) and metric entropy. First, we present the proof of Theorem 3.1, and then we prove some inequalities involving the local uniform dimensions of an invariant measure which are required for the other results. This section was inspired by [1, 48, 59].

Proof **(Theorem 3.1)**. Since the arguments used in the proof of the first and the last inequalities are similar, we just present the proof that, for each $q>1$, $D^-_\mu(q)\ge\alpha$. The second and the fourth inequalities come from Proposition I.2.

Fix $q>1$, let $x\in\operatorname{supp}(\mu)$, and let $\eta>0$; then, there exists an $\varepsilon(x)>0$ such that, for each $\varepsilon\in(0,\varepsilon(x))$ and each $y\in B(x,\varepsilon)$,

$$\frac{\log\mu(B(y,\varepsilon))}{\log\varepsilon}\ge\inf_{y\in\tilde{B}(x,\varepsilon)}\frac{\log\mu(B(y,\varepsilon))}{\log\varepsilon}\ge\alpha-\eta.$$

Thus, for each $x \in \text{supp}(\mu)$ and each $\eta > 0$, there exists an $\varepsilon(x) > 0$ such that, for each $\varepsilon \in (0, \varepsilon(x))$ and each $y \in B(x, \varepsilon)$,

$$\mu(B(y,\varepsilon)) \leq \varepsilon^{\alpha-\eta}. \tag{3.3}$$

Now, since $\{B(x, \varepsilon(x))\}_{x\in\text{supp}(\mu)}$ is an open covering of the compact set $\text{supp}(\mu)$, there exists a finite sub-family of $\{B(x, \varepsilon(x))\}_{x\in\text{supp}(\mu)}$ which also covers $\text{supp}(\mu)$. Let $\{B(x_i, \varepsilon(x_i))\}_{i=1}^{k}$ be this sub-covering and let $\varepsilon(k) := \min\{\varepsilon(x_1), \ldots, \varepsilon(x_k)\}$.

Consider the following (finite) covering of $\text{supp}(\mu)$ by balls of radius $\varepsilon(k)$:

$$\text{supp}(\mu) \subset \bigcup_{j=1}^{N} B(y_j, \varepsilon(k)),$$

where $y_j \in \overline{B}(x_l, \varepsilon(x_l))$ for some $l \in \{1, \ldots, k\}$ (note that since, for each $l \in \{1, \ldots, k\}$, $\overline{B}(x_l, \varepsilon(x_l))$ is compact, the open covering $\{B(y, \varepsilon(k))\}_{\{y\in\overline{B}(x_l,\varepsilon(x_l))\}}$ of $\overline{B}(x_l, \varepsilon(x_l))$ admits a finite sub-covering). Now, let $\{A_j\}_{j=1}^{M}$ be the disjoint covering of $\text{supp}(\mu)$ obtained by removing the self-intersections of the elements of the previous covering; then,

$$\text{supp}(\mu) = \biguplus_{j=1}^{M} A_j \cap \text{supp}(\mu). \tag{3.4}$$

Fix $j \in \{1, \ldots, M\}$ and let $y \in A_j \cap \text{supp}(\mu)$; there exists an $l \in \{1, \ldots, k\}$ such that $y \in B(x_l, \varepsilon(x_l)) \cap \text{supp}(\mu)$. It follows from (3.3) that, for each $0 < \varepsilon \leq \varepsilon(k) \leq \varepsilon(x_i)$, one has

$$\mu(B(y,\varepsilon)) \leq \varepsilon^{\alpha-\eta}.$$

Therefore,

$$\begin{aligned} \int_{A_j} \mu(B(y,\varepsilon))^{q-1} d\mu(y) &= \int_{A_j\cap\text{supp}\,\mu} \mu(B(y,\varepsilon))^{q-1} d\mu(y) \\ &\leq \int_{A_j\cap\text{supp}\,\mu} \varepsilon^{(q-1)(\alpha-\eta)} d\mu(y) = \varepsilon^{(q-1)(\alpha-\eta)}\, \mu(A_j). \end{aligned} \tag{3.5}$$

Now, by (3.4) and (3.5), one gets

$$\begin{aligned}\int_{\mathrm{supp}(\mu)} \mu(B(y,\varepsilon))^{q-1} d\mu(y) &= \int_{\biguplus_{j=1}^{M} A_j \cap \mathrm{supp}\,\mu} \mu(B(y,\varepsilon))^{q-1} d\mu(y)\\ &= \sum_{j=1}^{M} \int_{A_j \cap \mathrm{supp}\,\mu} \mu(B(y,\varepsilon))^{q-1} d\mu(y)\\ &\leq \sum_{j=1}^{M} \varepsilon^{(q-1)(\alpha-\eta)} \mu(A_j)\\ &= \varepsilon^{(q-1)(\alpha-\eta)}.\end{aligned}$$

Thus,

$$D_\mu^-(q) = \liminf_{\varepsilon\to 0} \frac{\log \int_{\mathrm{supp}(\mu)} \mu(B(y,\varepsilon))^{q-1} d\mu(y)}{(q-1)\log\varepsilon} \geq \alpha - \eta.$$

The result now follows, since $\eta > 0$ is arbitrary. □

Lemma 3.1. *Let (X, f, μ) be a dynamical system such that X is a Polish metric space and $\mu \in \mathcal{M}(f)$.*

i) *If f is a continuous function for which there exist constants $\Lambda > 1$ and $\delta > 0$ such that, for each $x, y \in X$ so that $d(x,y) < \delta$, $d(f(x), f(y)) \leq \Lambda\, d(x,y)$, then for each $x \in X$,*

$$\underline{d}_{\mu,i}(x) \geq \frac{\underline{h}_\mu(f,x)}{\log\Lambda}. \tag{3.6}$$

Moreover, if $\mu \in \mathcal{M}_e(f)$, it follows that

$$\dim_H^-(\mu) \geq \frac{h_\mu(f)}{\log\Lambda}. \tag{3.7}$$

ii) *If f is a continuous function for which if there exist constants $\lambda > 1$ and $\delta > 0$ such that, for each $x, y \in X$ so that $d(x,y) < \delta$, $\lambda\, d(x,y) \leq d(f(x), f(y))$, then for each $x \in X$,*

$$\overline{d}_{\mu,s}(x) \leq \frac{\overline{h}_\mu(f,x)}{\log\lambda}. \tag{3.8}$$

Moreover, if X is compact and $\mu \in \mathcal{M}_e(f)$, it follows that

$$\dim_P^+(\mu) \le \frac{h_\mu(f)}{\log \lambda}, \tag{3.9}$$

Here, $\overline{\underline{h}}_\mu(f,x) := \lim_{\varepsilon\to 0} \limsup(\inf)_{n\to\infty} \dfrac{-\log \mu(B(x,n,\varepsilon))}{n}$ is the upper (lower) local entropy of (X,f,μ) at x.

Proof. **i)** Claim 1. One has, for each $x \in X$, each $n \in \mathbb{N}$ and each $0 < \varepsilon \le \min\{1/2, \delta/2\}$, $B(x, \varepsilon\Lambda^{-n}) \subset B(x,n,\varepsilon)$, where $B(x,n,\varepsilon) := \{y \in X \mid d(f^i(x), f^i(y)) < \varepsilon,\ \forall\, i = 0,\dots,n\}$ is the Bowen ball of size n and radius ε, centered at x. Namely, fix $x \in X$, $n \in \mathbb{N}$ and $0 < \varepsilon \le \min\{1/2, \delta/2\}$, and let $y \in B(x, \varepsilon\Lambda^{-n})$; then, since $\varepsilon\Lambda^{-n} < \delta$, one has, for each $i = 0,\dots,n$, $d(f^i(x), f^i(y)) < \varepsilon$, proving the claim.

Now, it follows from Claim 1 that, for each $y \in \tilde{B}(x, \frac{\varepsilon\Lambda^{-n}}{2})$ and each $n \in \mathbb{N}$, $B(y, \frac{\varepsilon\Lambda^{-n}}{2}) \subset B(x, \varepsilon\Lambda^{-n}) \subset B(x,n,\varepsilon)$. Then,

$$\begin{aligned}
\underline{d}_{\mu,i}(x) &= \liminf_{n\to\infty} \inf_{y\in\tilde{B}(x,\frac{\varepsilon\Lambda^{-n}}{2})} \frac{\log \mu(B(y, \frac{\varepsilon\Lambda^{-n}}{2}))}{\log \frac{\varepsilon\Lambda^{-n}}{2}} \\
&\ge \liminf_{n\to\infty} \frac{\log \mu(B(x,n,\varepsilon))}{-n} \frac{1}{\frac{-\log\varepsilon}{n} + \log\Lambda + \frac{\log 2}{n}} \\
&\ge \liminf_{n\to\infty} \frac{\log \mu(B(x,n,\varepsilon))}{-n} \frac{1}{\log\Lambda}.
\end{aligned}$$

Thus, the result follows by taking $\varepsilon \to 0$ in both sides of the inequalities above.

Now, if $\mu \in \mathcal{M}_e(f)$, it follows from Lemma 2.8 in [44] that $\underline{h}_\mu(f,x) = \mu\text{-}\operatorname{ess\,inf} h_\mu(T,y)$ is valid for μ-a.e. x, and then, by Theorem 2.9 in [44], that $\underline{h}_\mu(f,x) \ge h_\mu(T)$ is also valid for μ-a.e. x. Relation (3.7) is now a consequence of relation (3.6) and Definition I.7.

ii) Claim 2. One has, for each $x \in X$, each $n \in \mathbb{N}$ and each $0 < \varepsilon \le \delta$, $B(x,n,\varepsilon) \subset B(x, \varepsilon\lambda^{-n})$. Namely, fix $x \in X$, $n \ge 1$ and $0 < \varepsilon \le \delta$, and let $y \in B(x,n,\varepsilon)$ so that, for each $j = 0,\dots,n$, $d(f^j(x), f^j(y)) < \varepsilon \le \delta$; it follows from the hypothesis that $\lambda^n\, d(x,y) \le d(f^n(x), f^n(y)) < \varepsilon$, and therefore that $d(x,y) < \varepsilon\lambda^{-n}$.

Now, it follows from Claim 2 that, for each $y \in \tilde{B}(x, \varepsilon\lambda^{-n})$ and each $n \in \mathbb{N}$, $B(x, n, \varepsilon) \subset B(x, \varepsilon\lambda^{-n}) \subset B(y, 2\varepsilon\lambda^{-n})$. Then,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\log \mu(B(x,n,\varepsilon))}{-n} \frac{1}{\log \lambda} &= \limsup_{n\to\infty} \frac{\log \mu(B(x,n,\varepsilon))}{-n} \frac{1}{\frac{-\log \varepsilon}{n} + \log \lambda - \frac{\log 2}{n}} \\
&\geq \limsup_{n\to\infty} \sup_{y \in \tilde{B}(x,\varepsilon\lambda^{-n})} \frac{\log \mu(B(y, 2\varepsilon\lambda^{-n}))}{\log 2\varepsilon\lambda^{-n}} \\
&= \overline{d}_{\mu,s}(x).
\end{aligned}$$

Thus, taking $\varepsilon \to 0$ in both side of the inequalities above, the result follows.

Now, if $\mu \in \mathcal{M}_e(f)$, it follows from Brin-Katok's Theorem that, for μ-a.e. $x \in X$, $\overline{h}_\mu(f, x) = \underline{h}_\mu(f, x) = h_\mu(f)$. Relation (3.9) is now a consequence of relation (3.8) and Definition I.7. □

Remark 3.1. It is straightforward to prove that the local uniform dimensions of a Borel probability measure coincide with their regular local dimensions:

$$\underline{d}_\mu(x) := \liminf_{\varepsilon\to 0} \frac{\log \mu(B(x,\varepsilon))}{\log \varepsilon} \tag{3.10}$$

and

$$\overline{d}_\mu(x) := \limsup_{\varepsilon\to 0} \frac{\log \mu(B(x,\varepsilon))}{\log \varepsilon}, \tag{3.11}$$

where $x \in X$ (if $x \notin \operatorname{supp} \mu$, then $\overline{\underline{d}}_\mu(x) := \infty$ also coincide with the respective local uniform dimensions). The reason that we deal with local uniform dimensions of invariant measures instead of the regular ones will become clear in the proof of Theorem 3.2.

Remark 3.2. We note that Brin-Katok's Theorem is pointwise satisfied for f-homogeneous measures: one has, for each $x \in X$,

$$\lim_{\varepsilon\to 0} \liminf_{n\to 0} \frac{-\log \mu(B(x,n,\varepsilon))}{n} = \lim_{\varepsilon\to 0} \limsup_{n\to 0} \frac{-\log \mu(B(x,n,\varepsilon))}{n} = h_\mu(f).$$

Proof. By the definition of a homogeneous measure, for each $\varepsilon > 0$, there exist $0 < \delta(\varepsilon) < \varepsilon$

and $c > 0$ such that, for each $n \in \mathbb{N}$ and each $x, y \in X$,

$$\mu(B(y, n, \delta(\varepsilon))) \leq c\, \mu(B(x, n, \varepsilon)).$$

Thus,

$$\lim_{\varepsilon \to 0} \limsup(\inf)_{n\to\infty} -\frac{1}{n} \log \mu(B(y, n, \delta(\varepsilon))) \geq \lim_{\varepsilon \to 0} \limsup(\inf)_{n\to\infty} -\frac{1}{n} \log \mu(B(x, n, \varepsilon)).$$

Analogously, for each $\tilde{\varepsilon} > 0$, there exist $0 < \tilde{\delta}(\tilde{\varepsilon}) < \tilde{\varepsilon}$ and $\tilde{c} > 0$ such that

$$\lim_{\tilde{\varepsilon} \to 0} \limsup(\inf)_{n\to\infty} -\frac{1}{n} \log \mu(B(x, n, \tilde{\delta}(\tilde{\varepsilon}))) \geq \lim_{\tilde{\varepsilon} \to 0} \limsup(\inf)_{n\to\infty} -\frac{1}{n} \log \mu(B(x, n, \tilde{\varepsilon})).$$

This proves that the limits do not depend on $x \in X$. The result follows now from Brin-Katok's Theorem. □

Corollary 3.2. *Let (X, f, μ) be a dynamical system such that μ is an f-homogeneous measure and f is a function which satisfies the hypothesis of Lemma 3.1. Then, for each $s < 1$ and each $q > 1$, one has*

$$\frac{h_\mu(f)}{\log \Lambda} \leq D_\mu^-(q) \leq D_\mu^-(1) \leq D_\mu^+(1) \leq D_\mu^+(s) \leq \frac{h_\mu(f)}{\log \lambda}.$$

Proof. It follows from the f-homogeneity of μ, Lemma 3.1 and Remark 3.2 that, for each $x \in X$,

$$\frac{h_\mu(f)}{\log \Lambda} \leq \underline{d}_{\mu,i}(x) \leq \overline{d}_{\mu,i}(x) \leq \frac{h_\mu(f)}{\log \lambda}.$$

The result is now a consequence of Theorem 3.1. □

Remark 3.3. The Bowen-Margulis measure is an example of measure that does not belong to set CD_0 in Theorem 3.6. In fact, one has $D_\mu(q) = \dim_H(\mu) = \dim_H(X)$ (which is equal 2 when f is Anosov).

3.2 Proofs of Proposition 3.1 and Theorem 3.4

Let (X, d) be a compact metric space, and let $f : X \to X$ be a continuous transformation. For each $n \in \mathbb{N}$, one defines a new metric d_n on X by the law

$$d_n(x, y) = \max\{d(f^k(x), f^k(y)) : k = 0, \cdots, n-1\}.$$

Note that, for each $\varepsilon > 0$, the open ball of radius ε centerded at $x \in X$ with respect to d_n coincides with the Bowen dynamical ball of size n and radius $\varepsilon > 0$, centered at x:

$$B(x, n, \varepsilon) = \{y \in X : d_n(x, y) < \varepsilon\}.$$

Proposition 3.2. *The metrics d_n and d induce the same topology on X.*

Proof. This is a direct consequence of the fact that f is a homeomorphism. □

Thus, for each $x \in X$, each $n \in \mathbb{N}$ and each $r > 0$, $B(x, n, r)$ is an open set

***Proof* (Proposition 3.1).**

Claim 1. For each $x \in X$, one has $\underline{d}_{\mu,i}(x) \geq h_\mu(T)\left[\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right]$.

We follow the proof of part 1 of Lemma 3.2 in [59]. Namely, let

$$\Lambda = \{x \in M \mid x \text{ is regular in the sense of Oseledec-Pesin} \\ \text{and } \lim_{\varepsilon \to 0} \liminf_{n_1, n_2 \to \infty} \frac{-\log \mu(B(x, n_1, n_2, \varepsilon))}{n_1 + n_2} = h_\mu(T)\},$$

where $B(x, n_1, n_2, \varepsilon) := \{y \in X \mid d(T^j x, T^j y) < \varepsilon,\ j = -n_2, \ldots, n_2\}$ is the bilateral Bowen ball of size $n_1 + n_2 + 1$ and radius ε.

Since μ is an f-homogeneus measure and T is a uniform hyperbolic transformation (note that the discussion presented in Remark 3.2 can be adapted to bilateral Bowen balls), it follows that $\Lambda = M$ (see [50, 52]). Let $\chi_i = e^{\lambda_i}$. For each $x \in M$ and each $\varepsilon > 0$, it is straightforward to show (as in [59]) that

$$\underline{d}_{\mu,i}(x) \geq (h_\mu(T) - \varepsilon) \left[\frac{1}{\log \frac{\chi_1 + 2\varepsilon}{1-\varepsilon}} + \frac{1}{\log \frac{\chi_2^{-1} + 2\varepsilon}{1-\varepsilon}} \right];$$

indeed, it is possible to show that, for each $\rho > 0$ and each $y \in B(x, K(x)^{-1}\rho/2)$, one has

$$B(y, K(x)^{-1}\rho/2) \subset B(x, K(x)^{-1}\rho) \subset B(x, n_1(\rho), n_2(\rho), \rho),$$

where $K(x) : M \to \mathbb{R}$ is a function that relates the distance in the x-chart and the Riemannian metric on M by the formula $\| \cdot - \cdot \|_x \leq K(x) d(\cdot, \cdot)$. Since $\varepsilon > 0$ is arbitrary, the claim follows.

Claim 2. For each $x \in X$, one has $\overline{d}_{\mu,i}(x) \leq h_\mu(T) \left[\frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right]$.

We follow the proof of part 2 of Lemma 3.2 in [59]. Namely, since X is a uniformly hyperbolic set, one may define $\phi : X \to \mathbb{R}$ by the law

$$\phi(x) = \phi = A_1 K_1 \min\{(\chi_1 + 2\varepsilon)^{-1}), (\chi_2^{-1} + 2\varepsilon)^{-1}\},$$

where $A_1 := \inf_{x \in X} A(x) > 0$ and $K_1 := \sup_{x \in X} K(x) < \infty$ (see the proof of Lemma 3.2 in [59] for details).

Now, since μ is f-homogeneous, it follows from Manẽ's estimate that, for each $x \in X$,

$$\limsup_{n_1, n_2 \to \infty} -\frac{1}{n_1 + n_2} \log(\mu(B(x, n_1, n_2, \phi))) \leq h_\mu(T).$$

The rest of the proof follows the same steps presented in the proof of Lemma 3.2 in [59], taking into account that $\Lambda_1 = X$.

The result follows now from Claims 1, 2, Theorem 3.1 and Proposition I.2 (for the case $q = 1$). □

Remark 3.4. As in Theorem 4.4 in [59], one has, for each $q \in \mathbb{R}$, that

$$D_\mu^\pm(q) = \underline{C}(\mu) = \overline{C}(\mu) = \underline{C}_L(\mu) = \overline{C}_L(\mu) = \underline{R}(\mu) = \overline{R}(\mu) = h_\mu(T)\left[\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right],$$

where $\underline{C}(\mu), \overline{C}(\mu), \underline{C}_L(\mu), \overline{C}_L(\mu)$ are the capacities and $\underline{R}(\mu), \overline{R}(\mu)$ are the upper and lower Renyi dimensions of μ.

***Proof* (Theorem 3.4)**. It suffices, from Proposition I.2, to prove the result for $q = 1$. It follows from Theorem 3.2 that there exist a hyperbolic metric d which induces an equivalent topology on X, and numbers $k > 1$, $\varepsilon > 0$ such that f is expansive under this metric and, for each $0 < r < \varepsilon/k$ and each $x \in X$, $B(x, n, r) \subset B(x, k^{-n}r)$. Thus,

$$\frac{\int \log \mu(B(x, k^{-n}r))d\mu(x)}{\log k^{-n}r} \leq \frac{\int \log \mu(B(x, n, r))d\mu(x)}{\log k^{-n}r}. \tag{3.12}$$

Claim.

$$\limsup_{n\to\infty} \frac{\int \log \mu(B(x, n, r))d\mu(x)}{\log k^{-n}r} \leq h_\mu(f) \log k.$$

Following the proof of Brin-Katok's Theorem, fix $r > 0$ and consider a finite measurable partition ξ such that $\operatorname{diam} \xi = \max_{C\in\xi} \operatorname{diam}(C) < r$. Let $\xi(x)$ be the element of ξ such that $x \in \xi(x)$, and let $C_n^\xi(x)$ be the element of the partition $\xi_n = \bigvee_{i=-n}^{n} f^{-i}\xi$ such that $x \in C_n^\xi(x)$. Given that $\xi(x) \subset B(x, r)$, one has

$$C_n^\xi(x) = \bigcap_{i=-n}^{n} f^{-i}(\xi(f^i x)) \subset \bigcap_{i=-n}^{n} f^{-i}(B(f^i x, r)) = B(x, n, r),$$

from which follows that

$$\frac{\int \log \mu(B(x, n, r))d\mu(x)}{-n} \leq \frac{\int \log \mu(C_n^\xi(x))d\mu(x)}{-n} = \frac{H(\xi_n)}{n},$$

where $H(\xi_n) = -\sum_{C_n^\xi(x)\in\xi_n} \mu(C_n^\xi(x)) \log \mu(C_n^\xi(x)) = \int -\log \mu(C_n^\xi(x))d\mu(x)$. Thus,

$$\limsup_{n\to\infty} \frac{\int \log \mu(B(x, n, r))d\mu(x)}{-n} \leq \limsup_{n\to\infty} \frac{H(\xi_n)}{n} = H(f, \xi) \leq h_\mu(f),$$

proving the claim.

Now, since for each $r > 0$, each $k > 1$ and each $n \in \mathbb{N}$, $\int \log \mu(B(x, k^{-n}r))d\mu(x)$ is finite (by (3.12) and Lemma 2.12 in [57]), it follows from an adaptation of Lemma A.6 in [35] that

$$\limsup_{r \to 0} \frac{\int \log \mu(B(x,r))d\mu(x)}{\log r} = \limsup_{n \to \infty} \frac{\int \log \mu(B(x, k^{-n}r))d\mu(x)}{\log k^{-n}r}. \tag{3.13}$$

One concludes the proof of the proposition combining relations (3.12) and (3.13) with Claim. □

CONCLUSIONS

1) By Theorem 1.1, one concludes that, typically, invariant measures associated with the full-shift system T in a product space $X = M^{\mathbb{Z}}$, whose alphabet M is uncountable (M is a perfect and separable metric space), have zero Hausdorff dimension, zero lower rate of recurrence, zero lower quantitative waiting time indicator, infinite packing dimension, infinite upper rate of recurrence and infinite upper quantitative waiting time indicator, with $\text{supp}(\mu) = M$.

   This implies that a typical measure is supported on set Z that is totally disconnected (given that $\dim_{top}(Z) = \dim_H(Z) = 0$); even more, one may take Z as a dense G_δ subset of X. Moreover, given $x \in Z$, there exists a time sequence (time scale) for which the first incidence of $O(x)$ (the orbit of x) to one of its spherical neighborhoods (which depend on time) occurs as fast as possible (that is, it is of order 1; this means that the first return time to those neighborhoods increases subpolynomially fast); accordingly, there exists a time sequence for which the first incidence of $O(x)$ to one of its spherical neighborhoods increases as fast as possible (that is, super-polynomially fast). One also concludes that almost every T-orbit $O(x)$ densely fills the whole space.

2) By Theorem 2.1 and Corollary 2.1, one concludes that if a topological dynamical system has a dense set of periodic measures (this is true if the systems satisfies the specification property; see Appendix B), then typically an invariant measure has, for each $q > 0$, zero lower q-generalized fractal dimension. This implies, in particular, that a typical invariant measure has zero upper Hausdorff dimension and zero lower rate of recurrence.

   Again, as in 1), one concludes that a typical measure is supported on a subset Z of X

that is totally disconnected and dense. Furthermore, given $x \in Z$, the first incidence of $O(x)$ to one of its spherical neighborhoods occurs as fast as possible (that is, the first return time to those neighborhoods increases subpolynomially fast).

3) By Theorem 2.2, one concludes that for the full-shift system (X, T) (where $X = M^{\mathbb{Z}}$ is endowed with a sub-exponential metric and the alphabet M is a perfect and compact metric space), a typical invariant measure has, for each $q > 1$, infinite upper q-correlation dimension. Under the same conditions, a typical invariant measure has, for each $s \in (0, 1)$ and each $q > 1$, zero lower s-generalized and infinite upper q-generalized dimensions.

   This implies that the orbit of a point $x \in Z$ has a very complex structure, being "tight" for some spatial scale, and spreading fast throughout the space for another scale.

4) One concludes, by Theorem 3.1 and Lemma 3.1, that given a topological dynamical system (X, f, μ), where f is Lipshitz and μ is an ergodic hyperbolic f-homogeneous measure with positive metric entropy, its lower q-correlation dimension is positive, for each $q \in \mathbb{R}$.

5) Proposition 3.1 states that an ergodic hyperbolic f-homogeneous measure associated with a $C^{1+\delta}$-Axiom A diffeomorphism ($\delta > 0$) over a two-dimensional Riemannian manifold, $f : M \to M$, has correlation dimension equal to $h_\mu(f)[1/\lambda_1 - 1/\lambda_2]$ (where λ_1 and λ_2 are the Lyapunov exponents of μ).

6) Theorem 3.6 states that if T is a C^1-Axiom A, then $\{\mu \in \mathcal{M}(T) \mid D_\mu^+(q) = 0, q > 1\}$ is a residual subset of $\mathcal{M}(T)$. Furthermore, by Corollary 3.1, if there exist a hyperbolic metric d compatible with the topology of X and $\mu \in \mathcal{M}(f)$ is such that $D_\mu^+(q) > 0$, then $h(f) \geq h_\mu(f) > 0$ (that is, the system is chaotic in this metric).

We can even say that the positivity of these fractal dimensions (that is, Hausdorff, packing, correlation dimensions) indicates that the system has a kind of chaotic behaviour, something that is related to the positivity of the topological entropy. Nonetheless, for the full-shift system over $X = \mathbb{R}^{\mathbb{Z}}$ (where $\{\mu \in \mathcal{M}(T) \mid h_\mu(T) = 0\}$ is a residual subset of

$\mathcal{M}(T)$), we can conclude that the chaotic behaviour is somewhat mild in the invariant sets where such measures are supported.

Certainly, fixed a dynamical system (like the full-shift over an uncountable alphabet or more general topological dynamical systems), we have seen that the study of generic dimensional (and of the rates of recurrence) properties of invariant measures has posed a different (and more refined) way to understand the behavior of typical orbits (with respect to such invariant measures).

In this setting, we would like to answer the following questions. Fixed a Borel probability measure defined on a Polish metric space (with certain dimensional and recurrence properties), for what kind of transformations such is an invariant measure? Is this true for a generic set of transformations over this space?

Appendix A

Proof of Proposition I.1

***Proof* (Proposition I.1).** Since the arguments in both proofs (for Hausdorff and packing dimensions) are similar, we just prove the statement for $\dim_P^+(\mu)$ and $\dim_P^-(\mu)$.

a) $\dim_P^+(\mu) = \mu\text{-}\operatorname{ess\,sup} \overline{d}_\mu(x)$. Let $\alpha \geq 0$. We show that if $\mu\text{-}\operatorname{ess\,sup} \overline{d}_\mu(x) \leq \alpha$, then $\dim_P^+(\mu) \leq \alpha$. In fact, since $\mu\text{-}\operatorname{ess\,sup} \overline{d}_\mu(x) = \inf\{a \in \mathbb{R} \mid \mu(\{x \mid \overline{d}_\mu(x) \leq a\}) = 1\} \leq \alpha$, one has $\mu(\{x \in X \mid \overline{d}_\mu(x) \leq \alpha\}) = 1$. It follows from the Definition I.7 that $\dim_P^+(\mu) \leq \dim_P(\{x \in X \mid \overline{d}_\mu(x) \leq \alpha\})$. Now, by Corollary 3.20(a) in [14], one has $\dim_P(\{x \in X \mid \overline{d}_\mu(x) \leq \alpha\}) \leq \alpha$. Thus, $\dim_P^+(\mu) \leq \alpha$.

Conversely, we show that if $\dim_P^+(\mu) \leq \alpha$, then $\mu\text{-}\operatorname{ess\,sup} \overline{d}_\mu(x) \leq \alpha$. Suppose that there exists $\delta > 0$ such that $\mu\text{-}\operatorname{ess\,sup} \overline{d}_\mu(x) \geq \alpha + \delta$; then, by the definition of essential supremum of a measurable function, there exists $E \in \mathcal{B}$, with $\mu(E) > 0$, such that for each $x \in E$, $\overline{d}_\mu(x) \geq \alpha + \delta/2$. Then, by Corollary 3.20(b) in [14], $\dim_P(E) \geq \alpha + \delta/2$, and therefore, $\dim_P^+(\mu) \geq \alpha + \delta/2$. This contradiction shows that $\mu\text{-}\operatorname{ess\,sup} \overline{d}_\mu(x) \leq \alpha$.

b) $\dim_P^-(\mu) = \mu\text{-}\operatorname{ess\,inf} \overline{d}_\mu(x)$. Let $\alpha > 0$. We show that if $\mu\text{-}\operatorname{ess\,inf} \overline{d}_\mu(x) \geq \alpha$, then $\dim_P^-(\mu) \geq \alpha$. By the definition of essential infimun of a measurable function, $\mu(A) = 1$, where $A := \{x \in X \mid \overline{d}_\mu(x) \geq \alpha\}$. Since, for each $E \in \mathcal{B}$, $\mu(E) = \mu(A \cap E)$ ($E \setminus A \subset A^c$), one may only consider, without loss of generality, those sets $E \in \mathcal{B}$ such that $E \subset A$. Thus, for each $A \supset E \in \mathcal{B}$ so that $\mu(E) > 0$, it follows from Corollary 3.20(b) in [14] that

$\dim_P(E) \geq \alpha$. The result is now a consequence of Definition I.7.

Conversely, we show that if $\dim_P^-(\mu) \geq \alpha$, then μ- $\operatorname{ess\,inf} \overline{d}_\mu(x) \geq \alpha$. Suppose that there exists $\delta > 0$ such that μ- $\operatorname{ess\,inf} \overline{d}_\mu(x) \leq \alpha - \delta$; then, by the definition of essential infimum of a measurable function, there exists $E \in \mathcal{B}$, with $\mu(E) > 0$, such that for each $x \in E$, $\overline{d}_\mu(x) \leq \alpha - \delta/2$. Thus, $E \subset \{x \in X \mid \overline{d}_\mu(x) \leq \alpha - \delta/2\} = C$ and $\dim_P E \leq \dim_P C$. Then, by Corollary 3.20(a) in [14], $\dim_P(E) \leq \alpha - \delta/2$, and therefore, $\dim_P^-(\mu) \leq \alpha - \delta/2$. This contradiction shows that μ- $\operatorname{ess\,sup} \overline{d}_\mu(x) \geq \alpha$. $\square$

Appendix B

Specification property

The specification property is defined as follows:

Definition B.1 (see [53])**.** One says that a homeomorphism $f : X \to X$ has the specification property (abbreviated to f satisfies specification) if for each $\varepsilon > 0$, there exists an integer $m = m(\varepsilon)$ such that the following holds: if

a) $I_1, \ldots, I_k$ are intervals of integers, $I_j \subseteq [a, b]$ for some $a, b \in \mathbb{Z}$ and all j,

b) $\operatorname{dist}(I_i, I_j) \geqslant m(\varepsilon)$ for $i \neq j$,

then for arbitrary $x_1, \ldots, x_k \in X$, there exists a point $x \in X$ such that

1) $f^{b-a+m}(x) = x$,

2) $d\left(f^n(x), f^n\left(x_j\right)\right) < \varepsilon$ for $n \in I_j$.

One can interpret this definition as follows: given $\varepsilon > 0$ and any finite number of pieces of orbits, sufficiently separated in time, one can find a periodic point, which ε-shadows the specified pieces of orbits.

Appendix C

Entropy in dynamical systems

Topological entropy

Let (X, d) be a compact metric space, and let $f : X \to X$ be a continuous transformation.

Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Recall that a subset F of X is said to be an (n, ε)-generating if, for each $x \in X$, there exists $y \in F$ such that $d_n(x, y) < \varepsilon$.

Let $R(n, \varepsilon)$ be the smallest cardinality of an (n, ε)-generating set for X with respect to f. Then, the following limit exists, and it is called the *topological entropy* of f (see [58]):

$$h(f) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log R(n, \varepsilon) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log R(n, \varepsilon). \tag{C.1}$$

Another equivalent approach consists in considering an (n, ε)-separated set. A set $\emptyset \neq E \subset X$ is called an (n, ε)-separated set if, for each $x, y \in E$, there exists an $0 \leq i < n$ such that $d_n(x, y) > \varepsilon$. Let $S(n, \varepsilon)$ be the maximal cardinality of an (n, ε)-separated set. Then (see [58]),

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S(n, \varepsilon) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log S(n, \varepsilon). \tag{C.2}$$

Metric entropy

Definition C.1 (Metric entropy)**.** Let $(X, \mathcal{B}(X), f, \mu)$ be a dynamical system (that is, $\mu \in \mathcal{M}(f)$). One defines the entropy of a finite partition, $\mathcal{Q}$, of X by the law

$$H_\mu(\mathcal{Q}) = -\sum_{E \in \mathcal{Q}} \mu(E) \log \mu(E).$$

If $\mathcal{P}$ and $\mathcal{Q}$ are two partitions of X, set $\mathcal{P} \vee \mathcal{Q} := \{F \cap E \mid F \in \mathcal{P} \text{ and } E \in \mathcal{Q}\}$. The metric entropy of $(X, \mathcal{B}, f, \mu)$ relative to the partition $\mathcal{Q}$ is defined by the law

$$h_\mu(f, \mathcal{Q}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{Q} \vee f^{-1}\mathcal{Q} \vee \cdots \vee f^{-n+1}\mathcal{Q}).$$

Finally, one defines the entropy of the dynamical system $(X, \mathcal{B}(X), f, \mu)$ by

$$h_\mu(f) = \sup h_\mu(f, \mathcal{Q}),$$

where the supremum is taken over all finite partitions, $\mathcal{Q}$, of X.

Definition C.2. Consider a compact metric space (X, d). Let $f : X \to X$ be a continuous map and μ an invariant Borel measure. We define the lower (upper) local (pointwise) entropies as follows:

$$\begin{aligned} \underline{h}_\mu(f, x) &= \lim_{\varepsilon\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu\left(B(x, \varepsilon, n)\right) \\ \overline{h}_\mu(f, x) &= \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu\left(B(x, \varepsilon, n)\right) \end{aligned}.$$

Note that the limits in ε exist due to monotonicity. We say that the local entropy exists at x if

$$h_\mu(f, x) = \underline{h}_\mu(f, x) = \overline{h}_\mu(f, x).$$

The following result gives an alternative representation of the metric entropy for some parti-cular dynamical systems on compact metric spaces. It is considered the topological version of Shannon-McMillan-Breiman's Theorem.

Theorem C.1 (**Brin-Katok** [10]). *Let (X, d) be a compact metric space, let $f : X \to X$ be a continuous function, and let $\mu \in \mathcal{M}(T)$ be non-atomic. Then, for μ-a.e. $x \in X$*

1) $h_\mu(f, x) = \underline{h}_\mu(f, x) = \overline{h}_\mu(f, x)$,

2) $h_\mu(f, x)$ *is an f-invariant function and*

3) $\int h_\mu(f, x)d\mu = h_\mu(f)$.

Since for ergodic dynamical systems, invariant functions are constant a.e., one has the follo-wing corollary of Brin-Katok's Theorem.

Corollary C.1. *If $(X, \mathcal{B}(X), f, \mu)$ is an ergodic dynamical system, then $h_\mu(f, x) = h_\mu(f)$ for μ-a.e. $x \in X$.*

The following definition and results are generalizations of the Brin-Katok Theorem for complete and separable metric spaces.

Definition C.3. The lower and upper local entropies of f relative to μ, denoted respectively by $\underline{h}_\mu^{loc}(f)$ and $\overline{h}_\mu^{loc}(f)$, are defined as

$$\underline{h}_\mu^{loc}(f) = \mu\text{-}\operatorname{ess\,inf} \lim_{\varepsilon \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \mu\left(B(x, \varepsilon, n)\right) = \mu\text{-}\operatorname{ess\,inf}\ \underline{h}_\mu(f, x)$$

and

$$\overline{h}_\mu^{loc}(f) = \mu\text{-}\operatorname{ess\,sup} \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu\left(B(x, \varepsilon, n)\right) = \mu\text{-}\operatorname{ess\,sup}\ \overline{h}_\mu(f, x).$$

Lemma C.1 (Lemma 2.8 in [44]). *Let X be a complete and separable metric space and let $f : X \to X$ be a continuous transformation. If μ is an ergodic f-invariant measure on X, then*

$$\underline{h}_\mu^{\text{loc}}(f) = \int \underline{h}_\mu(f, x)d\mu(x) \quad \text{and} \quad \overline{h}_\mu^{loc}(f) = \int \overline{h}_\mu(f, x)d\mu(x).$$

Theorem C.2 (Theorem 2.9 in [44]). *Let X be a complete and separable metric space and let $f : X \to X$ be a continuous transformation. If μ is an ergodic f-invariant measure on X, then*

$$h_\mu(f) \leqslant \underline{h}_\mu^{\text{loc}}(f).$$

The follow result is a study of Poincaré recurrence from a purely geometrical viewpoint. As we saw, Brin-Katok Theorem show that the metric entropy is given by the exponential growth rate of return times to dynamical balls. Varandas, in [56], showed that minimal return times to dynamical balls grow linearly with respect to its length and gave a relation with local entropy.

Definition C.4. Let X a compact metric space and let $f : X \to X$ be a continuous transformation. The *nth return time* $R_n(x,\varepsilon)$ *to the dynamical ball* $B(x,\varepsilon,n)$ is defined by

$$R_n(x,\varepsilon) = \inf \left\{ k \geq 1 \mid f^k(x) \in B(x,\varepsilon,n) \right\}.$$

Theorem C.3 (Theorem A in [56]). *Let μ be an ergodic f-invariant measure. The limits*

$$\overline{h}(f,x) = \lim_{\varepsilon \to 0} \limsup_{n\to\infty} \frac{1}{n} \log R_n(x,\varepsilon) \quad \text{and} \quad \underline{h}(f,x) = \lim_{\varepsilon \to 0} \liminf_{n\to\infty} \frac{1}{n} \log R_n(x,\varepsilon)$$

exist for μ-almost every x and coincide with the metric entropy $h_\mu(f)$.

Proposition C.1 (Proposition A in [56]). *Assume that $f : X \to X$ is a continuous transformation and that there exist constants $\delta, \lambda, \Lambda > 0$ such that $\lambda d(x,y) \leq d(f(x), f(y)) \leq \Lambda d(x,y)$ for every $x, y \in X$ so that $d(x,y) < \delta$. If μ is an f-invariant ergodic measure with positive entropy then*

$$\frac{h_\mu(f)}{\log \Lambda} \leq \underline{R}(x) \quad \text{and} \quad \overline{R}(x) \leq \frac{h_\mu(f)}{\log \lambda}$$

for μ-a.e. x.

Appendix D

Some generic results for M-valued discrete stochastic processes known in the literature

The following results are well-known in the literature; some of them are valid for more general systems (such those satisfying the specification property; see Definition B.1), but here we have opted to present them only for M-valued discrete stochastic processes.

Theorem D.1 (see [15, 50, 53])**.** *Let* $(X, \mathcal{B}, T)$ *be the full-shift system over* $X = \prod_{-\infty}^{+\infty} M$*, where the alphabet* M *is a separable metric space. Then:*

I. *if* $X = \mathbb{R}^{\mathbb{Z}}$*, then the set of invariant measures such that* $h_\mu(T) = 0$ *is a dense* G_δ *subset of* $\mathcal{M}(T)$ (see [51])*;*

II. $\mathcal{M}_e(T)$ *is residual in* $\mathcal{M}(T)$ (see [38, 39])*;*

III. *the set of invariant measures with full support is a dense* G_δ *subset of* $\mathcal{M}(T)$*;*

IV. *if* M *is also perfect and compact, then the set of invariant measures that are non-atomic is residual in* $\mathcal{M}(T)$*;*

V. *the set of invariant measures such that* $\mu(F) = 0$, *for each* $F \in \mathcal{B}(X)$ *which is* T*-invariant,* $\mathrm{int}(F) = \emptyset$ *and* $\overline{F} \subsetneq X$*, is a residual set of* $\mathcal{M}(T)$.

We present a proof of Theorem D.1-(III, IV and V). We begin showing that, for a Polish space (M, d) (that is, for a complete separable metric space), the set of Borel probability measures over M, whose respective supports contain a given closed set, is a G_δ (see [15, 50, 53]).

Proposition D.1. *Let* $\emptyset \neq F \subset M$ *be an arbitrary closed set. Then,*

$$B_F = \{\mu \in \mathcal{M} : F \subset \mathrm{supp}(\mu)\}$$

is a G_δ *set in* $\mathcal{M}$.

Proof. Let (x_i) be a countable dense set in F. Then,

$$B_F = \bigcap_i \{\mu \in \mathcal{M} : x_i \in \mathrm{supp}(\mu)\},$$

so it is sufficient to consider the case $F = \{x\}$. Since

$$\{\mu \in \mathcal{M} \mid x \notin \mathrm{supp}(\mu)\} = \bigcup_{m=1}^{\infty} \{\mu \in \mathcal{M} \mid \mu(B(x, 1/m)) = 0\}$$

(recall that $x \in \mathrm{supp}(\mu)$ if, and only if, for each $m \in \mathbb{N}$, $\mu(B(x, 1/m)) > 0$), one needs to show that the set $\mathcal{M}_{m,x} := \{\mu \in \mathcal{M} \mid \mu(B(x, 1/m)) = 0\}$ is closed.

Thus, let (μ_n) be a sequence of measures in $\mathcal{M}_{m,x}$ such that $\mu_n \to \mu$. Since, for each $n \in \mathbb{N}$, $\mu_n(B(x, 1/m)) = 0$, it follows from the definition of weak convergence that $0 = \liminf_{n\to\infty} \mu_n(B(x, 1/m)) \geq \mu(B(x, 1/m))$. Therefore, $\mu \in \mathcal{M}$. □

The next result shows that the set of T-invariant measures whose topological supports are equal to the closure of the T-periodic points (a point $x \in M$ is periodic if there exists a minimal positive integer k_x, the so-called period of x under T, such that $T^{k_x} x = x$),

$\overline{\mathcal{P}} \subset X$, is a G_δ-dense set in $\mathcal{M}(T)$ (in this setting, $\mu(\overline{\mathcal{P}}) = 1$ for each $\mu \in \mathcal{M}(T)$; see Theorems 3.3 and 4.1 in [39]).

Proposition D.2 (see [15, 50, 53])**.** *The set*

$$C_{\overline{\mathcal{P}}} = \{\mu \in \mathcal{M}(T) \mid \operatorname{supp}(\mu) = \overline{\mathcal{P}}\}$$

is dense in $\mathcal{M}$*.*

Proof. It is a direct consequence of Theorems 3.3 and 4.1 in [39] that $\{\mu \in \mathcal{M}(T) \mid \operatorname{supp}(\mu) \subset \overline{P}\} = \mathcal{M}(T)$; it follows from this statement and Proposition D.1 that it is sufficient to show that the set $B_{\overline{\mathcal{P}}} = \{\mu \in \mathcal{M}(T) \mid \operatorname{supp}(\mu) \supset \overline{P}\}$ is dense in $\mathcal{M}(T)$.

So, let $\{x_l\}_{l\geq 1}$ be a dense subset of $\mathcal{P}$, and let, for each $l \in \mathbb{N}$, $\lambda_l(\cdot) := \frac{1}{k_{x_l}} \sum_{i=0}^{k_{x_l}-1} \delta_{T^i x_l}(\cdot)$ be the periodic measure associated with x_l (see Lemma 4.1 in [39]). If this sequence of periodic measures is not dense in $\mathcal{M}(T)$, then one just has to add to it another dense enumerable sequence of periodic measures, say $\{\nu_l\}$ (see Theorems 1-(2.2) in [38] and Theorem 3.3 in [39] for a proof of the existence of such set). Now, let $\{\mu_n\} = \{\lambda_l\} \cup \{\nu_m\}$, and let $\{y_p\}$ be the respective countable dense set of periodic points associated to all of μ_n.

Since $\{\mu_n\}$ is dense in $\mathcal{M}(T)$, it is sufficient to show that given $\mu \in \{\mu_n\}$ and $\varepsilon > 0$, there exists $\nu \in B_{\overline{\mathcal{P}}}$ such that $\nu \in B(\mu, \varepsilon)$, where $B(\mu, \varepsilon)$ is the open ball of radius ε centered at μ (recall that $\mathcal{M}(T)$, endowed with the weak topology, is metrizable).

Hence, let $\eta := \sum_{j\geq 1} \frac{1}{2^j} \mu_j$; note that η is a T-invariant probability measure such that $\operatorname{supp}(\eta) = \overline{\mathcal{P}}$ (that is, $\eta \in C_{\overline{\mathcal{P}}}$). Now, define $\nu := (1 - \frac{\varepsilon}{2})\mu + \frac{\varepsilon}{2}\eta$. Then, for each $A \in \mathcal{B}$, one has $|\nu(A) - \mu(A)| \leq \varepsilon$, so ν belongs to the strong ε-neighborhood of μ (and therefore, to $B(\mu, \varepsilon)$).

Obviously, ν is a T-invariant probability measure; thus, one just needs to check that $\operatorname{supp}(\nu) = \overline{\mathcal{P}}$ in order to conclude that $\nu \in B_{\overline{\mathcal{P}}}$. But then, it is sufficient to show that if $p \in \{y_n\}$, then $p \in \operatorname{supp}(\nu)$.

Let μ_p be the periodic measure associated with p. It is clear that, for each $r > 0$,

$\mu_p(B(p,r)) > 0$; then, for each $r > 0$,

$$\begin{aligned}\nu(B(p,r)) = (1-\frac{\varepsilon}{2})\mu(B(p,r)) + \frac{\varepsilon}{2}\eta(B(p,r)) &\geq \frac{\varepsilon}{2}\eta(B(p,r)) \\ &\geq \frac{\varepsilon}{k_p\, 2^{k_p+1}}\mu_p(B(p,r)) > 0,\end{aligned}$$

where k_p is the period of p. Thus, $p \in \operatorname{supp}(\nu)$, which concludes the proof that $B_{\overline{P}}$ is dense in $\mathcal{M}(T)$. □

Proof **(Theorem D.1)**. **III**. It follows from Propositions D.1 and D.2. □

Proof **(Theorem D.1)**. **IV.** We follow the same ideas presented in the proof of Theorem 2 in [50]. For each $a > 0$, let $C(a)$ denote the set of measures $\mu \in \mathcal{M}(T)$ such that $\mu(\{x\}) \geq \tau$ for some $x \in X$.

(a) $C(a)$ *is closed.* Suppose that μ_n is a sequence in $C(a)$ which converges to $\mu \in \mathcal{M}(T)$. For each n, there exists an $x_n \in X$ such that $\mu_n(x_n) \geq a$. Since X is compact, there exists a subsequence of x_n that converges to $x_0 \in X$. So, for each fixed $\varepsilon > 0$, it follows from the definition of weak convergence that $\mu(\overline{B}(x_0,\varepsilon)) \geq \limsup \mu_n(\overline{B}(x_0,\varepsilon)) \geq a$. Since $\varepsilon > 0$ is arbitrary, one concludes that $\mu(\{x_0\}) \geq a$, and therefore, that $\mu \in C(a)$.

(b) $C(a)$ *is nowhere dense.* Suppose, by absurd, that there exists an open set $V \subset C(a)$. Let $p > 0$ be such that $1/p < a$. By Lemma 1.6, there exists a periodic point x with period $\tau(x) \geq p$ such that $\mu_x \in V$. Since $\mu_x(T^i x) \leq 1/p < a$, it follows that $\mu_x \notin C(a)$, which contradics the hypothesis.

Thus, $\bigcup_{n\in\mathbb{N}} C(1/n)$ is a set of first category in $\mathcal{M}(T)$. Its complement, that is, the set of nonatomic invariant measures, is a dense G_δ set in $\mathcal{M}(T)$. □

Proof **(Theorem D.1)**. **V.** Let $\mathcal{N} := \{\mu \in \mathcal{M}(T) \mid \mu(F) = 0$ for each $F \in \mathcal{B}(X)$ which is T-invariant, $\operatorname{int} F = \emptyset$ and $\overline{F} \subsetneq X\}$.

Since $C_X(T) \cap \mathcal{M}_e(T)$ is a dense G_δ set in $\mathcal{M}(T)$, it is sufficient to show that $C_X(T) \cap \mathcal{M}_e(T) \subset \mathcal{N}$. Let $\mu \in C_X(T) \cap \mathcal{M}_e(T)$, and let F be as in the definition of $\mathcal{N}$. Since $X \setminus \overline{F} \neq \emptyset$ is open and $\operatorname{supp}(\mu) = X$, it follows that $\mu(X \setminus \overline{F}) > 0$ (every $\mu \in C_X(T)$ gives positive weight to any open set). Now, since μ is ergodic and $X \setminus \overline{F}$ is T-invariant (given that T and T^{-1} are continuous, $\overline{F}$ is T-invariant), one has $\mu(X \setminus \overline{F}) = 1$, and therefore, $\mu(F) \leq \mu(\overline{F}) = 0$. □

A direct consequence of Theorem D.1-IV is:

Corollary D.1. *Let $S = \{F \subset X \mid F$ is closed and T-invariant$\}$, the set of subsystems of (X, T). Then, the subsystems $F \in S$ such that $\mathrm{int}(F) = \emptyset$ do not form a support set (that is, $\mu(F) = 0$) for typical measures. Furthermore, if X is a compact metric space, then all the proper subsystem $F \in S$ do not form a support set for typical measures.*

Proof. The first part follows by Theorem D.1-IV. The second part follows from the fact that the system is topologically transitive. □

Appendix E

Double sequences

Definition E.1. A double sequence of complex numbers is a function $s : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$. We shall use the notation $\{s(n,m)\}$. We say that a double sequence $\{s(n,m)\}$ converges to $a \in \mathbb{C}$, and we write $\lim_{n,m\to\infty} s(n,m) = a$, if the following condition is satisfied: for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that, for each $n, m > N$,

$$|s(n,m) - a| < \varepsilon.$$

The number a is called the double limit of the double sequence $\{s(n,m)\}$. If no such a exists, we say that the sequence $\{s(n,m)\}$ diverges.

Definition E.2. For a double sequence $\{s(n,m)\}$, the limits

$$\lim_{n\to\infty}(\lim_{m\to\infty} s(n,m)) \quad \text{and} \quad \lim_{m\to\infty}(\lim_{n\to\infty} s(n,m))$$

are called iterated limits.

Theorem E.1 (Theorem 2.13 in [23])**.** *Let $\{s(n,m)\}$ be a double sequence of natural numbers such that $\lim_{n,m\to\infty} s(n,m) = a$. Then, the iterated limits*

$$\lim_{n\to\infty}(\lim_{m\to\infty} s(n,m)) \quad \textit{and} \quad \lim_{m\to\infty}(\lim_{n\to\infty} s(n,m))$$

exist and both are equal to a if, and only if:

1) *for each* $n \in \mathbb{N}$, $\lim_{m\to\infty} s(n,m)$ *exists, and*

2) *for each* $m \in \mathbb{N}$, $\lim_{n\to\infty} s(n,m)$ *exists.*

Theorem E.2 (Theorem 2.15 in [23])**.** *If* $\{s(n,m)\}$ *is a double sequence such that*

1) $\lim_{n\to\infty}(\lim_{m\to\infty} s(n,m)) = a$, *and*

2) $\lim_{n\to\infty} s(n,m)$ *exists uniformly in* $m \in \mathbb{N}$,

then $\lim_{n,m\to\infty} s(n,m) = a$.

Bibliography

[1] Afraimovich, V., Chazottes, J. R., and Saussol, B. Pointwise dimensions for Poincaré recurrences associated with maps and special flows. *Discrete Contin. Dyn. Syst. 9*, 2 (2003), 263–280.

[2] Bao, H. Dimension, recurrence via entropy and Lyapunov exponents for C^1 map with singularities. *Ergodic Theory Dynam. Systems 38*, 3 (2018), 801–831.

[3] Baraviera, A. T., Cioletti, L. M., Lopes, A. O., Mohr, J., and Souza, R. R. On the general one-dimensional XY model: positive and zero temperature, selection and non-selection. *Rev. Math. Phys. 23*, 10 (2011), 1063–1113.

[4] Barbaroux, J.-M., Germinet, F., and Tcheremchantsev, S. Generalized fractal dimensions: equivalences and basic properties. *J. Math. Pures Appl. (9) 80*, 10 (2001), 977–1012.

[5] Barreira, L., Pesin, Y., and Schmeling, J. On the pointwise dimension of hyperbolic measures: a proof of the Eckmann-Ruelle conjecture. *Electron. Res. Announc. Amer. Math. Soc. 2*, 1 (1996), 69–72.

[6] Barreira, L., and Saussol, B. Hausdorff dimension of measures via Poincaré recurrence. *Comm. Math. Phys. 219*, 2 (2001), 443–463.

[7] Barreira, L., and Saussol, B. Product structure of Poincaré recurrence. *Ergodic Theory Dynam. Systems 22*, 1 (2002), 33–61.

[8] Bogachev, V. I. *Measure theory. Vol. I, II.* Springer-Verlag, Berlin, 2007.

[9] Boshernitzan, M. D. Quantitative recurrence results. *Invent. Math. 113*, 3 (1993), 617–631.

[10] Brin, M., and Katok, A. On local entropy. In *Geometric dynamics (Rio de Janeiro, 1981)*, vol. 1007 of *Lecture Notes in Math.* Springer, Berlin, 1983, pp. 30–38.

[11] Carvalho, S. L., and Condori, A. Generalized fractal dimensions of invariant measures of full-shift systems over compact and perfect spaces: generic behavior. *Forum Math. 33*, 2 (2021a), 435–450.

[12] Carvalho, S. L., and Condori, A. Generic properties of invariant measures of full-shift systems over perfect polish metric spaces. *Stochastics and Dynamics* (2021b), https://doi.org/10.1142/S0219493721500404.

[13] Cutler, C. D. A review of the theory and estimation of fractal dimension. In *Dimension estimation and models*, vol. 1 of *Nonlinear Time Ser. Chaos*. World Sci. Publ., River Edge, NJ, 1993, pp. 1–107.

[14] Cutler, C. D. The density theorem and Hausdorff inequality for packing measure in general metric spaces. *Illinois J. Math. 39*, 4 (1995), 676–694.

[15] Denker, M., Grillenberger, C., and Sigmund, K. *Ergodic theory on compact spaces.* Lecture Notes in Mathematics, Vol. 527. Springer-Verlag, Berlin-New York, 1976.

[16] Eckmann, J.-P., and Ruelle, D. Addendum: "Ergodic theory of chaos and strange attractors". *Rev. Modern Phys. 57*, 4 (1985), 1115.

[17] Falconer, K. *Fractal geometry - Mathematical Foundations and Applications.* Wiley, 1990.

[18] Fathi, A. Expansiveness, hyperbolicity and Hausdorff dimension. *Comm. Math. Phys. 126*, 2 (1989), 249–262.

[19] Fu, X.-C., Lu, W., Ashwin, P., and Duan, J. Symbolic representations of iterated maps. *Topol. Methods Nonlinear Anal. 18*, 1 (2001), 119–147.

[20] Galatolo, S. Dimension via waiting time and recurrence. *Math. Res. Lett. 12*, 2-3 (2005), 377–386.

[21] Gelfert, K., and Kwietniak, D. On density of ergodic measures and generic points. *Ergodic Theory Dynam. Systems 38*, 5 (2018), 1745–1767.

[22] Grassberger, P., Procaccia, I., and Hentschel, H. On the characterization of chaotic motions. *Lect.Notes Physics 179* (1983), 212–221.

[23] Habil, E. Double sequences and double series. *IUG Journal of Natural Studies 14*, 1 (2006), 219–233.

[24] Hirayama, M. Periodic probability measures are dense in the set of invariant measures. *Discrete Contin. Dyn. Syst. 9*, 5 (2003), 1185–1192.

[25] Hu, X.-h., Li, B., and Xu, J. Metric theorem and Hausdorff dimension on recurrence rate of Laurent series. *Bull. Korean Math. Soc. 51*, 1 (2014), 157–171.

[26] Hurewicz, W., and Wallman, H. *Dimension Theory.* Princeton Mathematical Series, v. 4. Princeton University Press, Princeton, N. J., 1941.

[27] Kac, M. On the notion of recurrence in discrete stochastic processes. *Bull. Amer. Math. Soc. 53* (1947), 1002–1010.

[28] Kwietniak, D., Oprocha, P., and Rams, M. On entropy of dynamical systems with almost specification. *Israel J. Math. 213*, 1 (2016), 475–503.

[29] Ledrappier, F. Dimension of invariant measures. In *Proceedings of the conference on ergodic theory and related topics, II (Georgenthal, 1986)* (1987), vol. 94 of *Teubner-Texte Math.*, Teubner, Leipzig, pp. 116–124.

[30] Li, J., and Oprocha, P. Properties of invariant measures in dynamical systems with the shadowing property. *Ergodic Theory Dynam. Systems 38*, 6 (2018), 2257–2294.

[31] Liang, C., Liu, G., and Sun, W. Approximation properties on invariant measure and Oseledec splitting in non-uniformly hyperbolic systems. *Trans. Amer. Math. Soc. 361*, 3 (2009), 1543–1579.

[32] Lopes, A. O., Mengue, J. K., Mohr, J., and Souza, R. R. Entropy and variational principle for one-dimensional lattice systems with a general *a priori* probability: positive and zero temperature. *Ergodic Theory Dynam. Systems 35*, 6 (2015), 1925–1961.

[33] Mandelbrot, B. B. *The fractal geometry of nature.* W. H. Freeman and Co., San Francisco, Calif., 1982. Schriftenreihe für den Referenten. [Series for the Referee].

[34] Mattila, P., Morán, M., and Rey, J.-M. Dimension of a measure. *Studia Math. 142*, 3 (2000), 219–233.

[35] Moritz, A. An introduction to and comparison of three notions of dimension in metric spaces. *Master theses, Mathematisch Instituut of the Universiteit Leiden* (2013).

[36] Munroe, M. E. *Measure and integration*, second ed. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1971.

[37] Oxtoby, J. C. Ergodic sets. *Bull. Amer. Math. Soc. 58* (1952), 116–136.

[38] Oxtoby, J. C. On two theorems of Parthasarathy and Kakutani concerning the shift transformation. In *Ergodic Theory (Proc. Internat. Sympos., Tulane Univ., New Orleans, La., 1961).* Academic Press, New York, 1963, pp. 203–215.

[39] Parthasarathy, K. R. On the category of ergodic measures. *Illinois J. Math. 5* (1961), 648–656.

[40] Pesin, Y., and Tempelman, A. Correlation dimension of measures invariant under group actions. *Random Comput. Dynam. 3*, 3 (1995), 137–156.

[41] Pesin, Y. B. On rigorous mathematical definitions of correlation dimension and generalized spectrum for dimensions. *J. Statist. Phys. 71*, 3-4 (1993), 529–547.

[42] Pesin, Y. B. *Dimension theory in dynamical systems.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1997. Contemporary views and applications.

[43] Ren, X. Periodic measures are dense in invariant measures for residually finite amenable group actions with specification. *Discrete Contin. Dyn. Syst. 38*, 4 (2018), 1657–1667.

[44] Riquelme, F. Ruelle's inequality in negative curvature. *Discrete Contin. Dyn. Syst. 38*, 6 (2018), 2809–2825.

[45] Rogers, C. A. *Hausdorff measures.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. Reprint of the 1970 original, With a foreword by K. J. Falconer.

[46] Rudnicki, R. Pointwise dimensions and Rényi dimensions. *Proc. Amer. Math. Soc. 130*, 7 (2002), 1981–1982.

[47] Sadovskaya, V. Dimensional characteristics of invariant measures for circle diffeomorphisms. *Ergodic Theory Dynam. Systems 29*, 6 (2009), 1979–1992.

[48] Saussol, B. Recurrence rate in rapidly mixing dynamical systems. *Discrete Contin. Dyn. Syst. 15*, 1 (2006), 259–267.

[49] Saussol, B. An introduction to quantitative Poincaré recurrence in dynamical systems. *Rev. Math. Phys. 21*, 8 (2009), 949–979.

[50] Sigmund, K. Generic properties of invariant measures for Axiom A diffeomorphisms. *Invent. Math. 11* (1970), 99–109.

[51] Sigmund, K. On the prevalence of zero entropy. *Israel J. Math. 10* (1971), 281–288.

[52] Sigmund, K. On mixing measures for axiom A diffeomorphisms. *Proc. Amer. Math. Soc. 36* (1972), 497–504.

[53] Sigmund, K. On dynamical systems with the specification property. *Trans. Amer. Math. Soc. 190* (1974), 285–299.

[54] Simpelaere, D. Correlation dimension. *J. Statist. Phys. 90*, 1-2 (1998), 491–509.

[55] Smale, S. Differentiable dynamical systems. *Bull. Amer. Math. Soc. 73*, 6 (11 1967), 747–817.

[56] Varandas, P. Entropy and Poincaré recurrence from a geometrical viewpoint. *Nonlinearity 22*, 10 (2009), 2365–2375.

[57] Verbitskiy, E. Generalized entropies in dynamical systems. *PhD Thesis, University of Groningen/UMCG research database* (2000).

[58] Walters, P. *An introduction to ergodic theory*, vol. 79 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Berlin, 1982.

[59] Young, L. S. Dimension, entropy and Lyapunov exponents. *Ergodic Theory Dynamical Systems 2*, 1 (1982), 109–124.