UNIVERSIDAD NACIONAL DE INGENIERÍA

FACULTAD DE CIENCIAS

TESIS

"On the Kernel of the Gysin Homomorphism on Chow Groups of Zero Cycles and Applications"

PARA OBTENER EL GRADO ACADÉMICO DE DOCTOR EN CIENCIAS CON MENCIÓN EN MATEMÁTICA

ELABORADA POR:
RINA ROXANA PAUCAR ROJAS

ASESOR:
Dr. JOE ALBINO PALACIOS BALDEÓN

ASESORA EXTERNA:
Dra. CLAUDIA SCHOEMANN

LIMA-PERÚ

2023

Dedicatory

A quitter never wins..., and a winner never quits.
Napoleon Hill.

Make everything as simple as possible, but not simpler.
Albert Einstein.

Acknowledgement

There are many people I would like to thank, who have had significant positive influences on me during my PhD studies and throughout the development of this thesis.

I would like to deeply thank my advisors, Joe Palacios and Claudia Schoemann for their encouragement and support throughout the time of my PhD studies. Both their mathematical and practical advice have been invaluable for my development. It is hard to emphasize enough how much I appreciated their professional advice and patience in answering my questions.

I would like to thank my referees Dr. Vladimir Guletskiĭ and Prof. Dr. Daniel Huybrechts for careful reading of the thesis and advising me on how to improve it further and also for the suggestions for next phase of research.

I would like to express my sincere gratitude to Prof. Dr. Harald Helfgott, Prof. Dr. Matthias Schütt, Prof. Dr. Stefan Schreieder, and Prof. Dr. Daniel Huybrechts for their support to join to their research teams during my PhD studies which allowed me to gain skills and experience relevant to my field of study, and also for their kindness in answering my math questions concerning this PhD thesis.

I have made many friends throughout and prior to my PhD studies who have contributed greatly to this thesis. I would like to thank Reinder Meinsma, for being a wonderful officemate, for always being there to chat and especially for answer many of my math questions through his extreme care and precision. I thank to Qixiao Ma and Matthias Paulsen, for being very generous to engaging in extensive email correspondence regarding various results needed to proof the main result of this PhD thesis and for teaching me and outlining clearly the proofs of them, and also for their numerous helpful suggestions, enlightening conversations, and motivating talks. I would like also to thank Giacomo Mezzedimi, Erik Nikolov, Carolina Neira, Yulieth Prieto, Milagros Anculli, Yaxy Pintado, Patty arce, Álida Silva, Edith Anco, Maritza Moreno, Joe Lope, Hector Llanos, Eyüp Samet Balkan, Theodosis Alexandrou, Edgar Ayala Elvira, Karl Christ, Salvatore Floccari, Victor Gonzalez-Alonso, Morten Wolfram Arnold Lüders, Nebojsa Pavic, Emre Can Sertöz, Domenico Valloni, Sara Torelli, Evgeny Shinder, Yajnaseni Dutta, Yoonjoo Kim, Kuan-Wen Lai, Gebhard Martin, Dominique Mattei, Jiexiang Huang, Maximillian Schimpf, Mauro Varesco, Jiexiang Huang, Ronald Mas, David Saldaña, for their friendship, helpful comments and conversations. I thank all the other mathematicians and friends I have made prior and throughout graduate studies.

I thank my mother Maria Rojas Alarcon, my father Eugenio Paucar Duran, my brothers Eugenio Paucar Rojas and Braulio Sanchez Rojas, my sister Noemi Paucar Rojas and my niece Maryam Jurado Paucar for their continual love and support.

I would like to thank the "PROGRAMA DE DOCTORADO EN UNIVERSIDADES PERUANAS" CG-176-2015-FONDECYT for generously funding my PhD project.

Contents

Abstract

Let S be a connected smooth projective surface over $\mathbb{C}$. Let Σ be the complete linear system of a very ample divisor D on S and let $d = \dim(\Sigma)$. For any closed point $t \in \Sigma \cong \mathbb{P}^{d*}$, let H_t be the hyperplane in $\mathbb{P}^d$ corresponding to t, $C_t = H_t \cap S$ the corresponding hyperplane section of S, and r_t the closed embedding of C_t into S. Let Δ_S be the discriminant locus of Σ parametrizing singular hyperplane sections of S and $U = \Sigma \setminus \Delta_S$ its complement of smooth hyperplane sections of S. Let $\mathsf{CH}_0(S)_{\deg=0}$ and $\mathsf{CH}_0(C_t)_{\deg=0}$ be the Chow groups of 0-cycles of degree zero of S and C_t respectively. In this thesis we prove that for C_t a smooth hyperplane section of S the Gysin kernel, i.e., the kernel of the Gysin homomorphism from $\mathsf{CH}_0(C_t)_{\deg=0}$ to $\mathsf{CH}_0(S)_{\deg=0}$ induced by r_t, is a countable union of translates of an abelian subvariety A_t inside the Jacobian J_t of the curve C_t. Then we prove that there is a c-open subset U_0 in U such that $A_t = 0$, for all $t \in U_0$, or $A_t = B_t$, for all $t \in U_0$; where B_t is an abelian subvariety of J_t. Finally, we prove that if we are in the case where Δ_S is an hypersurface, then $A_t = 0$ or $A_t = B_t$, for every $t \in U$.

As an application of the main result of the thesis we prove a theorem on 0-cycles on surfaces and we study the connection of this theorem with Bloch's conjecture and constant cycles curves.

Resumen

Sea S una superficie suave, proyectiva y conexa sobre $\mathbb{C}$. Sea Σ el sistema lineal completo de un divisor muy amplio D en S y sea $d = \dim(\Sigma)$. Para cualquier punto cerrado $t \in \Sigma \cong \mathbb{P}^{d*}$, sea H_t el hiperplano en $\mathbb{P}^d$ correspondiente a t, $C_t = H_t \cap S$ la correspondiente sección hiperplana de S, y r_t el embebimiento cerrado de C_t en S. Sea Δ_S el lugar discriminante de Σ parametrizando secciones hiperplanas singulares de S y $U = \Sigma \setminus \Delta_S$ su complemento parametrizando secciones hiperplanas suaves de S. Sean $\mathsf{CH}_0(S)_{\mathrm{deg}=0}$ y $\mathsf{CH}_0(C_t)_{\mathrm{deg}=0}$ los grupos de Chow de 0-ciclos de grado cero en S y C_t respectivamente. En esta tesis probamos que para C_t una seccion hiperplana suave de S el Gysin kernel, i.e., el kernel del Gysin homomorfismo de $\mathsf{CH}_0(C_t)_{\mathrm{deg}=0}$ a $\mathsf{CH}_0(S)_{\mathrm{deg}=0}$ inducida por r_t, es una union contable de trasladados de una subvariedad abeliana A_t contenida en el Jacobiano J_t de la curva C_t. Luego probamos que existe un subconjunto c-abierto U_0 en U tal que $A_t = 0$, para todo $t \in U_0$, o $A_t = B_t$, para todo $t \in U_0$, donde B_t es una subvariedad abeliana de J_t. Finalmente, probamos que si estamos en el caso donde Δ_S es una hipersuperficie, para todo $t \in U$ tenemos que $A_t = 0$ o $A_t = B_t$.

Como una aplicación del resultado principal de la tesis probamos un teorema sobre 0-ciclos en superficies y estudiamos la conexión de este teorema con la conjetura de Bloch y con la noción de curvas ciclo constantes.

Introduction

Let k be an algebraically closed field of characteristic 0, let S be a smooth projective surface over k, let $\mathsf{CH}_0(S)_{\deg=0}$ be the Chow group 0-cycles of degree zero on S, let alb_S be the Albanese morphism defined from $\mathsf{CH}_0(S)_{\deg=0}$ to the Albanese variety $\mathrm{Alb}(S)$ of S. Bloch's conjecture states that if S has geometric genus zero, then alb_S is an isomorphism, see [7] and [5].

If the Kodaira dimension of S is < 2, i.e., S is of special type, Bloch's conjecture has been proven in [8]. If S has Kodaira dimension 2, i.e., S is of general type, the vanishing of the geometric genus of S implies the vanishing of the irregularity of S, and the conjecture simply states that any two points on S are rationally equivalent to each other. This is the hard case of Bloch's conjecture and only known for some particular cases.

In [1] Banerjee and Guletskiĭ show a general version on the countability results of the Gysin kernel related to the countability results of the Gysin kernel for surfaces stated in [30, pages 304-305]. They provide a formal and abstract proof based on the étale monodromy argument. Let us comment their approach. Let X be a smooth projective connected variety of dimension $2p$ embedded into $\mathbb{P}^m$ over an uncountable algebraically closed field k of characteristic 0, let Y a hyperplane section of X, and let $\mathsf{A}^p(Y) = \frac{\mathsf{Z}^p(Y)_{\mathsf{alg}}}{\mathsf{Z}^p(Y)_{\mathsf{rat}}}$ (resp. $\mathsf{A}^{p+1}(X) = \frac{\mathsf{Z}^{p+1}(X)_{\mathsf{alg}}}{\mathsf{Z}^{p+1}(X)_{\mathsf{rat}}}$) be the continuous part of the Chow group $\mathsf{CH}^p(Y)$ (resp. $\mathsf{CH}^{p+1}(X)$), that is, algebraically trivial algebraic cycles modulo rational equivalence on Y (resp. on X). Whenever Y is smooth and satisfying three assumptions (the group $\mathsf{A}^p(Y)$ is regularly parametrized by an abelian variety A, $\mathsf{A}^p(Y) = \mathsf{CH}^p(Y)_{\deg=0}$ and $H^1_{et}(A, \mathbb{Q}_l(1-p)) \cong H^{2p-1}_{et}(Y, \mathbb{Q}_l)$, see §2 in [1]) they prove that the kernel of the Gysin pushforward homomorphism from $\mathsf{A}^p(Y)$ to $\mathsf{A}^{p+1}(X)$ induced by the closed embedding of Y into X is the union of a countable collection of shifts of a certain abelian subvariety A_0 inside A, and for a very general section Y either $A_0 = 0$ or A_0 coincides with an abelian subvariety A_1 in A. Due to their assumptions the case $p = 1$ of this result gives an approach to prove Bloch conjecture.

On the other hand, the notion of constant cycles curves was introduced by Huybrechts on $K3$ surfaces, see [16]. There are various equivalent definitions of constant cycle curves, the interesting definition for us over $\mathbb{C}$, which is algebraically closed and uncountable, is that a curve $C \subset S$ is a constant cycle curve if and only if the map

r_{C*} from $\mathrm{Pic}^0(\tilde{C}) = \mathsf{CH}_0(\tilde{C})_{\deg=0}$ to $\mathsf{CH}_0(S)$ is the zero map, where $r_C : \tilde{C} \to S$ is the composition of the normalization $\tilde{C} \to C$ with the closed embedding $C \hookrightarrow S$. The most important examples of constant cycles curves are provided by rational curves, but not every constant cycle curve is rational so it is still not known how much weaker the notion of constant cycles curves is. Moreover, to find a criteria that decides whether a given curve is a constant cycle curve seems as hard as to find a criteria that would ensure the opposite (see [16, introduction]).

In this thesis we study the Gysin kernel for the case of surfaces and we prove some results on the countability of the Gysin kernel related to the countability results of the Gysin kernel stated in [30, pages 304-305] and [1] which play an important role in the study of 0-cycles on surfaces, especially in the context of Bloch's conjecture and constant cycle curves. More precisely, let S be a connected smooth projective surface over $\mathbb{C}$. Let Σ be the complete linear system of a very ample divisor D on S and let $d = \dim(\Sigma)$. For any closed point $t \in \Sigma \cong \mathbb{P}^{d*}$, let H_t be the hyperplane in $\mathbb{P}^d$ corresponding to t, $C_t = H_t \cap S$ the corresponding hyperplane section of S, and r_t the closed embedding of C_t into S. Let $U = \Sigma \setminus \Delta_S$ be the complement of the discriminant locus of Σ of smooth hyperplane sections of S. We prove that whenever C_t is a smooth hyperplane section the Gysin kernel G_t, i.e., the kernel of the Gysin homomorphism r_{t*} from $\mathsf{CH}_0(C_t)_{\deg=0}$ to $\mathsf{CH}_0(S)_{\deg=0}$ induced by r_t, is the union of a countable collection of translates of an abelian subvariety A_t inside $B_t \subset J_t$, where A_t is the unique irreducible component passing through zero of the irredundant decomposition of G_t, and B_t is the abelian subvariety of the Jacobian J_t of the curve C_t corresponding to the Hodge substructure on $H^1(C_t, \mathbb{Z})_{\mathsf{van}}$ (this result is the case $p = 1$ of Theorem A in [1] and it is also a proof of [30, page 304, Exercise 1 a]). Then, using the approach to prove Theorem B in [1], we prove that there is a c-open U_0 in U such that $A_t = 0$, for all $t \in U_0$, or $A_t = B_t$, for all $t \in U_0$; that is, we prove that for all $t \in U_0$ we have that A_t has only two possibilities and the same behaviour (using this result we prove that if the Gysin kernel G_t is countable for a very general section C_t then alb_S is an isomorphism which is useful criterion to prove Bloch's conjecture). This result is the case $p = 1$ of Theorem B in [1] which can be also compared with [30, page 304, Exercise 1 b]. Finally, we prove that if we are in the case where the discriminant locus Δ_S of Σ is an hypersurface, then $A_t = 0$ or $A_t = B_t$, for every $t \in U$, i.e., we prove that for every $t \in U$ we have that A_t has only two possibilities but not necessarily the same behaviour. This last result applied to surfaces with $q(S) = 0$ gives a criterion to determine if the corresponding smooth hyperplane section C_t is a constant cycle curve or not.

It is important to note that in order to prove the countability results on the Gysin kernel for the case of smooth projective connected variety of dimension $2p$ in [1] Banerjee and Guletskiĭ make three assumptions called Assumption 1, Assumption 2, and Assumption 3. In this thesis we also prove that these assumptions are not necessary for the case of surfaces because they turn out to be true facts which we prove and call Fact 1, Fact 2 and Fact 3 respectively.

The plan of the thesis is as follows. In §1 we give the necessary background about intersection theory. More precisely, we give a short introduction to the theory of algebraic cycles, rational equivalence, algebraic equivalence, homological equivalence, divisors and linear systems. In particular, this section provides the necessary background to prove *Fact 2* which gives us a lot of information about the ways we can think of the Chow group of 0-cycles of degree zero of the smooth hyperplane sections of a surface. We also prove the needed results of these topics which we could not find easily in other texts.

In §2 we recall some facts about Hodge theory, that is, we give a short introduction to the notion of Hodge structure, polarized varieties, the morphism of Hodge structures, the Abel-Jacobi map, the Albanese map and the Albanese variety. In particular, this section provides the necessary background to prove *Fact 1* and *Fact 3*.

In §3 we define the Lefschetz pencils of hyperplane sections on an n-dimensional smooth projective variety which is a notion that provides a qualitative description of the generators of the vanishing cohomology in degree $n-1$ of a smooth hyperplane section of the variety, that is, the kernel of the Gysin homomorphism on cohomology groups induced by the embedding of a smooth hyperplane section of the variety into the variety, in order to obtain this description it is important to consider a Lefschetz pencil of hyperplane sections of the variety passing through this hyperplane section. In this chapter we also study the monodromy action on the cohomology of the fibres of a projective morphism.

In §4 we prove the main result of the thesis called a *theorem on the Gysin kernel* (Theorem 4.1.1). More precisely, let S be a connected smooth projective surface over $\mathbb{C}$. Let Σ be the complete linear system of a very ample divisor D on S of dimension say d. Let Δ_S be the discriminant locus of Σ parametrizing singular hyperplane sections of S and $U = \Sigma \setminus \Delta_S$ its complement of smooth hyperplane sections of S. For any closed point $t \in \Sigma \cong \mathbb{P}^{d*}$, let H_t be the hyperplane in $\mathbb{P}^d$ corresponding to t, $C_t = H_t \cap S$ the corresponding hyperplane section of S, r_t the closed embedding of C_t into S, and r_{t*} the Gysin homomorphism on Chow groups from $\mathsf{CH}_0(C_t)_{\deg=0}$ to $\mathsf{CH}_0(S)_{\deg=0}$, induced by r_t, whose kernel is denoted by G_t and called the **Gysin kernel** associated to the hyperplane section C_t. Using properties of the Chow groups of 0-cycles of degree zero of the smooth hyperplane sections C_t of a surface S we prove in item a) of this theorem that the Gysin kernel G_t associated to a smooth hyperplane section C_t, i.e., with $t \in U$, is a countable union of translates of an abelian subvariety A_t inside the Jacobian J_t of the curve C_t. Then using the approach of [1] we prove in item b) that there exits a subset U_0 in U such that A_t with $t \in U_0$ has only two possibilities and also the same behaviour. Finally, using the argument of the monodromy we prove that if we are in the case where Δ_S is a hypersurface, for every t in U we have that A_t has only two possibilities but not necessarily the same behaviour.

In §5, as an application of Theorem 4.1.1 we prove Theorem 5.1.1 called a *theorem on the* 0*-cycles on surfaces* (see also Theorem C in [1]). This theorem states that if the Albanese morphism $alb_S : \mathsf{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S)$ is not an isomorphism, then G_t is countable for a very general t. In this chapter we also study the connection of Theorem 4.1.1 and Theorem 5.1.1 with Bloch's conjecture and constant cycles curves. More precisely, we show that item c) of Theorem 4.1.1 applied to surfaces with irregularity zero (see item c) of Corollary 5.3.6) gives a criteria to determine if the smooth curve C_t is a constant cycle curve or not. On the other hand, we show that the Theorem 5.1.1 or more precisely its contrapositive and hence equivalent form gives us a criteria to prove Bloch's conjecture (see Corollary 5.2.1) and this theorem applied to surfaces with $q(S) = 0$ gives us a criteria to prove Bloch's conjecture for surfaces of general type and with $p_g(S) = 0$ using the notion of constant cycles curves (see Corollary 5.3.7).

Chapter 1

Algebraic Cycles

The main purpose of this chapter is to recall some needed facts about a beautiful subject in algebraic geometry called *intersection theory*. This chapter begins with the definition of algebraic cycles, then we study some operations on algebraic cycles (the pushforward, pullback, intersection product, actions of a correspondence), next we define rational equivalence which allows us to study some properties of the Chow groups of r-cycles $\mathsf{CH}_r(X)$ on a scheme X and in particular of the Chow group $\mathsf{CH}_0(X)_{\mathrm{deg}=0}$ of 0-cycles of degree zero, which is the main mathematical object of this thesis, after that we study the notion of algebraic equivalence which allows us to define the continuous part $\mathsf{A}_r(X)$ of the Chow group, i.e., the group of r-cycles algebraically equivalent to zero modulo the group of r-cycles rationally equivalent to zero. We then continue with the definition of homological equivalence which allows us to define the group $\mathsf{CH}_r(X)_{\mathrm{hom}}$ of r-cycles homologically equivalent to zero modulo the group of r-cycles rationally equivalent to zero. Next we study the relationship between rational, algebraic and homological equivalence which allows to conclude that when X is a connected smooth projective variety over an algebraically close field of characteristic zero $\mathsf{CH}_0(X)_{\mathrm{deg}=0}$, $\mathsf{A}_0(X)$ and $\mathsf{CH}_0(X)_{\mathrm{hom}}$ are isomorphic to each other (see *Fact 2*), therefore we gain a lot of information to study $\mathsf{CH}_0(X)_{\mathrm{deg}=0}$ via these isomorphisms.

The last two subsections of this chapter are dedicated to recall briefly some facts about schemes, e.g., the notion of Weil divisors, Cartier divisors, we also see that there is a one to one correspondence between them when X is a nonsingular variety and if in addition they are effective we can think of them as subschemes of X, we then show that there is a one to one correspondence between Cartier divisors and some invertible sheaves, next we state some conditions on the invertible sheaves such that they provide closed embeddings to the projective space. We end this chapter with the definition of linear systems and its characterization as the set of closed points of a projective space.

We state and verify some important results needed for the proof of the main result. The topics in this chapter are likely well known to experts, but we present it also as a matter of fixing notation.

1.1 Algebraic cycles

Definition 1.1.1 (r-Cycle)**.** Let X be a scheme over k.

- An *algebraic cycle* on X is a formal finite linear combination

$$Z = \sum_i n_i Z_i,$$

where $n_i \in \mathbb{Z}$, and Z_i are subvarieties of X.

- If all the Z_i have the same dimension r we say that Z is an *algebraic cycle of dimension* r on X or simply an r*-cycle* on X.

The set

$$\mathsf{Z}_r(X) = \{Z = \sum_i n_i Z_i : Z_i \text{ a subvariety of dimension } r \text{ on X, } n_i \in \mathbb{Z}\}$$

of r-cycles on X is a free abelian group called the *group of r-cycles on X*.

Definition 1.1.2 (Purely dimensional scheme)**.** Let $X_1, \dots, X_t$ be the irreducible components of the scheme X. We say that X is *purely n-dimensional* if $\dim(X_i) = n$, for all i.

Remarks 1.1.3.

1. If X is a purely n-dimensional scheme we have

$$\mathsf{Z}_r(X) = \mathsf{Z}^{n-r}(X),$$

where $\mathsf{Z}^{n-r}(X)$ is the *group of algebraic cycles of codimension* $n - r$ on X (see [24, §1.1.]).

2. If we want to work with linear combinations in a field F, we write

$$\mathsf{Z}_r(X)_F = \mathsf{Z}_r(X) \otimes_{\mathbb{Z}} F,$$

(see [24, §1.1.]).

Example 1.1.4. Let X be a variety of dimension n.

1. 0-*cycles* on X are finite formal linear combination

$$Z = \sum_i n_i P_i,$$

where $n_i \in \mathbb{Z}$ and P_i are points of X.

2. *Cycles of codimension* 1 or $(n-1)$-*Cycles* or *divisors* on X are finite formal linear combination

$$Z = \sum_i n_i Z_i,$$

where $n_i \in \mathbb{Z}$ and Z_i are subvarieties of codimension 1 of X.

The following informal example will help us to gain more intuition about algebraic cycles

Example 1.1.5. Let k be an algebraically closed field. Let X be a smooth projective curve of degree 3 in the projective plane $\mathbb{P}^2_k$, then it looks like in the Figure 1.1

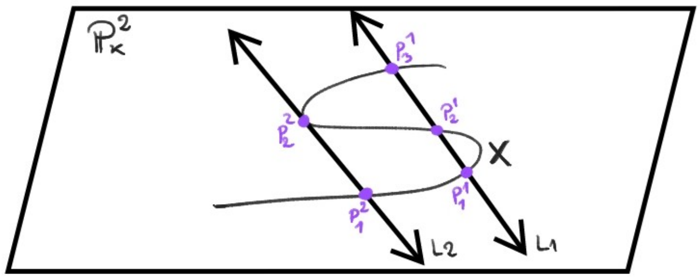

Figure 1.1: Algebraic cycles on a curve

Note that the intersection of the curve X with any line in the plane $\mathbb{P}^2_k$ is a finite set of points (see Figure 1.1). Since the curve is of degree 3 and if we count the points with its multiplicities, there are exactly 3 points in the intersection of the curve X with any line the plane $\mathbb{P}^2_k$.

- For the line L_1 in $\mathbb{P}^2_k$ of Figure 1.1, writing

$$Z_1 = P^1_1 + P^1_2 + P^1_3,$$

  where P^1_i, $i = 1, 2, 3$ are the points of the intersection of X and L_1 and the coefficients are the respective multiplicities of the points P^1_i, we get the 0-cycle Z_1 on the curve X.

- For the line L_2, proceeding in a similar way, we get the 0-cycle

$$Z_2 = P^2_1 + 2P^2_2.$$

Note also that since $\dim(X) = 1$, Z_1 and Z_2 are also *divisors* on the curve X, so as we vary the lines in $\mathbb{P}^2_k$ we obtain a family of divisors on X parametrized by lines in $\mathbb{P}^2_k$ (points of $\mathbb{P}^{2*}_k$). We call this set of divisors a *linear system of divisors on* X, which we will define later (this example is an adaptation of an example in [15, §6]).

Definition 1.1.6 (Geometric multiplicity)**.** Let X be a scheme or a subscheme of some scheme X', and let $X_1, \ldots, X_t$ be the irreducible components of X. The *geometric multiplicity* of X_i in X, is the length of the zero dimensional local rings $\mathcal{O}_{X_i,X}$, i.e,

$$m_i = l_{\mathcal{O}_{X_i,X}}(\mathcal{O}_{X_i,X}).$$

Definition 1.1.7 (Fundamental cycle)**.** Let X be a scheme or a subscheme of some scheme X'. The *fundamental cycle* of X is the cycle

$$[X] = \sum_{i=0}^{t} m_i X_i \in \mathsf{Z}(X),$$

where X_i are the irreducible components of X and m_i is the geometric multiplicity of X_i in X.

If X is purely r-dimensional, then $[X] \in \mathsf{Z}_r(X)$.

Pushforward of cycles

Definition 1.1.8 (The "intuitively" number of sheets)**.** Let $f : X \to Y$ be a proper morphism of schemes and $Z \subset X$ a subvariety of X. The *"intuitively" number of sheets* of Z over $f(Z)$, denoted by $\deg(Z/f(Z))$, is defined by

$$\deg(Z/f(Z)) = \begin{cases} [k(Z) : k(f(Z))] & \text{if } \dim f(Z) = \dim(Z) \\ 0 & \text{if } \dim f(Z) < \dim(Z). \end{cases}$$

where $[k(Z) : k(f(Z))]$ denotes the extension degree of the field extension.

The name of the above definition comes from the complex case, because in the complex case if $\dim(f(Z)) = \dim(Z)$, then Z is generically a covering of $f(Z)$ with $[k(Z) : k(f(Z))]$ sheets.

Definition 1.1.9 (Push-forward homomorphism)**.** Let $f : X \to Y$ be a proper morphism of schemes and $Z \subset X$ a subvariety of X.

- Set
$$f_*(Z) = \deg(Z/f(Z)) f(Z).$$
Note that, by definition, if Z is a subvariety of dimension r on X then $f(Z)$ is a subvariety in Y and $f_*(Z)$ is an r-cycle on Y.

- Extending linearly, one gets the *pushforward homomorphism*
$$f_* : \mathsf{Z}_r(X) \to \mathsf{Z}_r(Y),$$
induced by f, from the group of r-cycles on X to the group of r-cycles on Y.

Pullback of cycles

Definition 1.1.10 (Flat morphism)**.** A morphism of schemes $f : X \to Y$ is *flat* if one of the following equivalent facts are true:

1. for $V = \mathsf{Spec}(B) \subset Y$, $U = \mathsf{Spec}(A) \subset X$ affine open sets with $f(U) \subset V$, the induced map $B \to A$ makes A a flat module over B.

2. for all subvarieties W of X, $\mathcal{O}_{W,X}$ is a flat module over $\mathcal{O}_{\overline{f(W)},Y}$.

Definition 1.1.11 (Relative dimension of a morphism)**.** A morphism of schemes $f : X \to Y$ is of *relative dimension* n if for every subvariety V of Y and for every irreducible component Z of $f^{-1}(V)$ we have $\dim(Z) = \dim(V) + n$.

Definition 1.1.12 (Pullback of cycles)**.** Let $f : X \to Y$ be a flat morphism of schemes of relative dimension n, and let $V \subset Y$ be a subvariety of Y.

- Set
$$f^*(V) = [f^{-1}(V)],$$
where $f^{-1}(V)$ is the inverse image scheme which is a subscheme of X of pure dimension $\dim(V)+n$ (see [10, Appendix B.2.3]), and $[f^{-1}(V)]$ is the fundamental cycle of $f^{-1}(V)$.

- By linearity, this definition can be extended to the set of r-cycles and yields the *pullback homomorphisms*
$$f^* : \mathsf{Z}_r(Y) \to \mathsf{Z}_{r+n}(X).$$

If we restrict to the smooth projective reduced schemes we have the following alternative definition of pullback, see [24, chapter 1].

Definition 1.1.13 (Pullback of cycles)**.** Let $f : X \to Y$ be a morphism of smooth projective reduced schemes and let $Z \subset Y$ any subvariety. The graph of f is a subvariety $\Gamma_f \subset X \times Y$ and if it meets $X \times Z$ properly, we set
$$f^*(Z) := (\mathsf{pr}_X)_*(\Gamma_f \cdot (X \times Z)),$$
where $\mathsf{pr}_X : X \times Y \to X$ is the first projection and $\cdot$ denotes the intersection product which we define next.

Intersection product

In this subsection we define the intersection product for the case of dimensionally transverse intersections. For the general definition we refer to [10].

Let X be a smooth variety. Recall that two subvarieties V and W of X with codimension i and j intersect in a union of subvarieties Z_b of different codimensions $\geq i + j$.

Definition 1.1.14 (Dimensionally transverse/Proper intersections)**.** If two subvarieties V and W of X with codimension i and j respectively, intersect in a union of subvarieties Z_b of codimension $i + j$ we say that the intersection of V and W is *dimensionally transverse* or *proper*.

Definition 1.1.15 (Intersection multiplicity/Intersection number)**.** If V and W are two subvarieties of X whose intersection is dimensionally transverse and if Z_b is an irreducible component of $V \cap W$, then the *intersection multiplicity* or the *intersection number* of V and W along Z_b is a positive integer defined by

$$i(V \cdot W; Z_b) := \sum_r (-1)^r l_A(Tor_r^A(A/I(V), A/I(W))); A = \mathcal{O}_{X,Z_b},$$

where $I(V)$ (resp. $I(W)$) is the ideal of the variety V (resp. W) in the ring $A = \mathcal{O}_{X,Z_b}$.

Definition 1.1.16 (Intersection product or Intersection cycle class)**.** The *intersection product* of two subvarieties V and W whose intersection is dimensionally transverse is defined by

$$V \cdot W = \sum_b i(V \cdot W; Z_b) Z_b$$

where the sum runs over the irreducible components Z_b of $V \cap W$.

Remark 1.1.17. Note that the intersection product is an algebraic cycle.

Action of a correspondence on cycles

In this section all the varieties will be complete, i.e., proper over the ground field and nonsingular.

Definition 1.1.18 (Correspondence)**.** A *correspondence* from a variety X to a variety Y is a subvariety, a cycle or an equivalence class of cycles on $X \times Y$.

Definition 1.1.19 (Action of correspondences on cycles)**.** Let X be a variety of dimension n. A correspondence $W \in \mathsf{Z}^r(X \times Y)$ induces a homomorphism

$$\begin{array}{rccc} W: & \mathsf{Z}^i(X) & \to & \mathsf{Z}^{i+(r-n)}(Y) \\ & T & \mapsto & W(T) = [\mathsf{pr}_Y]_*(W \cdot (T \times Y)) \end{array}$$

and $(r-n)$ is called the *degree* of the correspondence.

Remark 1.1.20. If X is projective then the second projection $X \times Y \to Y$ is proper. Then a correspondence $Z \in \mathsf{CH}_r(X \times Y)$ defines a morphism

$$\begin{array}{rccc} Z_*: & \mathsf{CH}_l(X) & \to & \mathsf{CH}_{l+r-\dim(X)}(Y) \\ & Z' & \mapsto & Z_*(Z') = \mathsf{pr}_{2*}(\mathsf{pr}_1{}^*(Z') \cdot Z) \end{array}$$

1.2 Equivalence relations on algebraic cycles

Rational equivalence

Let X be a scheme.

Definition 1.2.1 (r-cycle associated to a rational function)**.** Let W be any subvariety of X of dimension $r+1$, and let $f \in k(W)^*$ be a nonzero rational function on W, then we can define a *r-cycle associated to f* on X as follows

$$div(f) = \sum_V \mathsf{ord}_V(f)V,$$

where V runs over all subvarieties of codimension 1 on W, and $\mathsf{ord}_V(f)$ is the order of vanishing of f along V, see [10, §1.2].

Definition 1.2.2 (r-cycle rationally equivalent to 0)**.** A r-cycle Z on X is *rationally equivalent to zero*, denoted by $Z \sim_{\mathsf{rat}} 0$, if there is a finite number of $(r+1)$-dimensional subvarieties W_i of X and $f_i \in k(W_i)^*$ such that

$$Z = \sum_i div(f_i).$$

The set of r-cycles rationally equivalent to 0 is denoted by

$$\mathsf{Z}_r(X)_{\mathsf{rat}} = \{Z \in \mathsf{Z}_r(X) : Z \sim_{\mathsf{rat}} 0\}.$$

It is a subgroup of $\mathsf{Z}_r(X)$.

Definition 1.2.3 (Rationally equivalent cycles)**.** Two r-cycles Z_1 and Z_2 on X are *rationally equivalent*, denoted by $Z_1 \sim_{\mathsf{rat}} Z_2$, if its difference $Z_1 - Z_2$ is rationally equivalent to 0.

Informally we say that two r-cycles Z and Z' on X are rationally equivalent if there exists a family of r-cycles on X parametrized by $\mathbb{P}^1$ interpolating between them. More precisely, if we restrict to smooth projective varieties we have the following alternative definition of rational equivalence, see for example [24, Lemma 1.2.5], [9, §1.2.2], [6, Introduction] and [14, Introduction].

Definition 1.2.4 (Rationally equivalent cycles)**.** Let X be a smooth projective variety. Two r-cycles Z and Z' on X are *rationally equivalent* if there exits $W \in \mathsf{Z}_{r+1}(X \times \mathbb{P}^1)$ such that for any $t \in \mathbb{P}^1$ defining by

$$W(t) := (\mathsf{pr}_X)_*(W \cdot (X \times \{t\})),$$

where $\cdot$ is the intersection product and pr_X is the projection to X, we have

$$Z = W(t_1) \text{ and } Z' = W(t_2) \text{ for some } t_1, t_2 \in \mathbb{P}^1.$$

Remark 1.2.5.

1. Note that $W(t)$ are the members of the family of r-cycles on X obtained by intersecting the $(r+1)$-cycle W with the fibers $X \times \{t\}$ over $t \in \mathbb{P}^1$, i.e., they are the fibers of W.

2. If in the above definition we set $Z' = 0$ we get the definition of a r-cycle rationally equivalent to 0.

In terms of codimension the definition is as follows

Definition 1.2.6 (Rationally equivalent cycles)**.** Let X be a smooth projective variety. Two cycles Z and Z' of codimension i on X are *rationally equivalent* if there exits $W \in \mathsf{Z}^i(X \times \mathbb{P}^1)$ such that for any $t \in \mathbb{P}^1$ defining by

$$W(t) := (\mathsf{pr}_X)_*(W \cdot (X \times \{t\})),$$

we have

$$Z = W(t_1) \text{ and } Z' = W(t_2) \text{ for some } t_1, t_2 \in \mathbb{P}^1.$$

Now we define the Chow groups.

Definition 1.2.7 (Chow groups)**.** Let X be a scheme.

The group quotient

$$\mathsf{CH}_r(X) = \frac{\mathsf{Z}_r(X)}{\mathsf{Z}_r(X)_{\mathsf{rat}}},$$

of rational equivalence classes of r-cycles is called the *Chow group of r-cycles on X*. If we work with coefficients in a field F we will write

$$\mathsf{CH}_r(X)_F = \mathsf{CH}_r(X) \otimes_{\mathbb{Z}} F.$$

Algebraic equivalence

Definition 1.2.8 (Algebraic equivalence)**.** Let X be a smooth projective reduced scheme. A cycle Z of codimension r on X is *algebraically equivalent to* 0, denoted by $Z \sim_{\mathsf{alg}} 0$, if and only if there exits a smooth connected curve C, a cycle $W \in \mathsf{Z}^r(X \times C)$ such that for any $t \in C$ defining by

$$W(t) := (\mathsf{pr}_X)_*(W \cdot (X \times \{t\})),$$

we have

$$W(t_1) = Z \text{ and } W(t_2) = 0$$

for some $t_1, t_2 \in C$.

The set of cycles of codimension r that are algebraically equivalent to 0 is denoted by

$$\mathsf{Z}^r(X)_{\mathsf{alg}} = \{Z \in \mathsf{Z}^r(X) : Z \sim_{\mathsf{alg}} 0\}.$$

It is a subgroup of $\mathsf{Z}^r(X)$.

If X is a complex smooth projective variety equivalently we can define algebraic equivalence as follows, see [30, §8.2.1], [7, Introduction].

Definition 1.2.9 (Cycle associated to an intersection)**.** Let C be a smooth connected curve and $W \subset C \times X$ a closed algebraic subset of codimension r of X each of whose components dominates C (i.e. the restriction to the components of the projection over C is dominant). Then, for each point $t \in C$ we can consider

$$W(t) = [W \cap (\{t\} \times X)],$$

the cycle of codimension r on X associated to the schematic intersection of W with the fiber $\{t\} \times X \simeq X$ over t via the first projection.

Remark 1.2.10. If we denote by $\tau = \mathsf{pr}_X|_W : W \to X$ and by $\pi = \mathsf{pr}_C|_W : W \to C$, where pr_X and pr_C are the projections to X and C respectively, then we can define $W(t)$ as follows

$$W(t) = \tau_* \circ \pi^*(t).$$

Definition 1.2.11 (Alternative definition of $\mathsf{Z}^r(X)_{\mathsf{alg}}$)**.** The subgroup $\mathsf{Z}^r(X)_{\mathsf{alg}}$ of cycles of codimension r algebraically equivalent to 0 is the subgroup generated by the cycles of codimension r the form $W(t) - W(t')$, for any smooth connected curve C, any points $t, t' \in C$, and for any cycle $W \in \mathsf{Z}^r(C \times X)$ each of whose components dominates C.

Now we can define the continuous part of the Chow group.

By definition it is clear that $\mathsf{Z}_r(X)_{\mathsf{rat}} \subset \mathsf{Z}_r(X)_{\mathsf{alg}}$, so we can define the following group quotient

Definition 1.2.12 (The continuous part of the Chow group)**.** The group quotient of cycles algebraically equivalent to 0 modulo rational equivalence is denoted by

$$\mathsf{A}_r(X) = \frac{\mathsf{Z}_r(X)_{\mathsf{alg}}}{\mathsf{Z}_r(X)_{\mathsf{rat}}} \subset \mathsf{CH}_r(X).$$

This group should be thought of as the continuous part of the Chow group of r-cycles (see [6, Introduction]).

Homological equivalence

Let X be a smooth projective variety over an algebraically closed field k.

Definition 1.2.13 (Weil-cohomology)**.** Fix a field F of characteristic 0 called the coefficient field. A *Weil-cohomology theory* is a contravariant functor

$$X \to H^*(X)$$

from the category of varieties to the category of augmented, finite dimensional, anti-commutative F-algebras which satisfies the following properties (see [17, §1.2.])

1. Poincaré duality: if $\dim(X) = n$, then

   a) The groups $H^r(X) = 0$, for $r \neq 0, \dots, 2n$.
   b) There is a given orientation isomorphism $H^{2n}(X) \simeq F$ (note that in particular, $H^0(P) \simeq F$, for P a point).
   c) The canonical pairings

   $$H^r(X) \times H^{2n-r}(X) \to H^{2n}(X)$$

   are non-singular.

   Let $H_r(X)$ be the F-vector space dual to $H^r(X)$. Then Poincaré duality states that there are isomorphisms

   $$H^{2n-r}(X) \xrightarrow{\sim} H_r(X)$$

   induced by the map $a \mapsto \langle \cdot, a \rangle$, where $\langle , \rangle : H^*(X) \to F$ is the degree map.

   Let $f : X \to Y$ be a morphism, $f^* = H^*(f) : H^*(Y) \to H^*(X)$ and define a F-linear map $f_* : H^*(X) \to H^*(Y)$ as the transpose of f^*. Then

   $$f_*((f^*a) \cdot b) = a \cdot f_*b.$$

2. Künneth formula: let $\mathsf{pr}_X : X \times Y \to X$ and $\mathsf{pr}_Y : X \times Y \to Y$ be the projections. Then

   $$\begin{array}{ccc} H(X) \otimes_F H(Y) & \to & H(X \times Y) \\ a \otimes b & \mapsto & \mathsf{pr}_X^*(a) \cdot \mathsf{pr}_Y^*(b) \end{array}$$

   is an isomorphism.

3. Cycle maps: there are groups homomorphisms

   $$cl_X : \mathsf{Z}^r(X) \to H^{2r}(X)$$

   satisfying the following properties

   - functorial in the sense that for a morphism of varieties $f : X \to Y$, one has

     $$f^* \circ cl_Y = cl_X \circ f^* \text{ and } f_* \circ cl_X = cl_Y \circ f_*.$$

   - Multiplicativity:

     $$cl_{X \times Y}(Z \times W) = cl_X(Z) \otimes cl_Y(W);$$

   - Non-triviality: if P is a point, then

     $$cl : \mathsf{Z}^*(P) = \mathbb{Z} \to H^*(P) = F$$

     is the canonical inclusion.

The elements of $H^*(X)$ are called *cohomology classes*, the multiplication on $H^*(X)$ is called *cup product*.

Remark 1.2.14. There are other more restrictive definitions of a Weil cohomology theory, see for example [24, Definition 1.2.13], where they define the Weil cohomology over the category of smooth projective reduced schemes over an arbitrary field k and they state that a Weil cohomology theory also satisfies the following properties:

a) There are *cycle class* maps

$$\mathrm{cl}_X : \mathsf{CH}^r(X) \to H^{2r}(X)$$

which are functorial, compatible with intersection product and compatible with points.

b) Weak Lefschetz holds: if $i : Y \hookrightarrow X$ is a smooth hyperplane section of a variety of dimension n, then

$$H^r(X) \xrightarrow{i^*} H^r(Y)$$

is an isomorphism for $r < n-1$ and injective for $r = n-1$.

c) Hard Lefschetz holds: the Lefschetz operator $L(\alpha)$ induces isomorphisms

$$L^{n-r} : H^{n-r}(X) \xrightarrow{\sim} H^{n+r}(X), 0 \leq r \leq n.$$

Fixing a Weil-cohomology theory we can now define the homological equivalence

Definition 1.2.15 (Homological equivalence)**.** A cycle Z of codimension r on X is *homologically equivalent to* 0, denoted by $Z \sim_{\text{hom}} 0$, if $cl_X(Z) = 0$.

This definition depends on the choice of a Weil cohomology theory ([24, Definition 1.2.16]).

The set of cycles of codimension r homologically equivalent to 0 form a group, it is denoted by $\mathsf{Z}^r(X)_{\text{hom}}$.

Let X be a smooth complex quasi-projective variety or more generally let X be a smooth projective reduced scheme over an arbitrary field k, fixing a Weil cohomology theory we have the following lemma

Lemma 1.2.16. *Let* $\dim(X) = n$ *and let* $cl_X(Z) \in H^{2n-2r}(X)$ *be the class of a* r*-cycle* Z *on* X*. If* $Z \sim_{\mathsf{rat}} 0$*, then* $cl_X(Z) = 0$*.*

Proof. For complex case see proof of [30, Lemma 9.18]. For the general case, note that this property form part of the properties of the Weil cohomology theory (see item a) of Remark 1.2.14). $\square$

By this lemma (i.e. the fact that $\mathsf{Z}_r(X)_{\mathsf{rat}} \subset \mathsf{Ker}(cl)$) and the fundamental theorem on homomorphisms, the cycle map

$$cl_X : \quad \mathsf{Z}_r(X) \quad \to \quad H^{2n-2r}(X)$$

thus gives the *cycle class map*

$$\mathrm{cl}_X : \mathsf{CH}_r(X) \to H^{2n-2r}(X).$$

Definition 1.2.17 (The group $\mathsf{CH}_r(X)_{\mathrm{hom}}$)**.** The kernel of the cycle *class* map cl_X is denoted by

$$\mathsf{CH}_r(X)_{\mathrm{hom}} = \mathsf{Ker}(\mathrm{cl}_X : \mathsf{CH}_r(X) \to H^{2n-2r}(X))$$

Note that $\mathsf{CH}_r(X)_{\mathrm{hom}}$ is the group of r-cycles homologically equivalent to 0 modulo rational equivalence, i.e.,

$$\mathsf{CH}_r(X)_{\mathrm{hom}} = \frac{\mathsf{Z}_r(X)_{\mathrm{hom}}}{\mathsf{Z}_r(X)_{\mathsf{rat}}}.$$

Relation between algebraic, rational and homological equivalence of algebraic cycles

Let X be a smooth projective reduced scheme over an algebraically closed field k of characteristic 0, then we have the following proposition.

Proposition 1.2.18.

$$\mathsf{Z}^r(X)_{\mathsf{rat}} \subset \mathsf{Z}^r(X)_{\mathsf{alg}} \subset \mathsf{Z}^r(X)_{\mathrm{hom}}.$$

Proof. The inclusion $\mathsf{Z}^r(X)_{\mathsf{rat}} \subset \mathsf{Z}^r(X)_{\mathsf{alg}}$ is clear by definition.

To prove the second inclusion $\mathsf{Z}^r(X)_{\mathsf{alg}} \subset \mathsf{Z}^r(X)_{\mathrm{hom}}$, assume that $Z \in \mathsf{Z}^r(X)_{\mathsf{alg}}$, that is, $Z \sim_{\mathsf{alg}} 0$, then by definition of algebraic equivalence there exits a connected smooth curve C, a cycle $W \in \mathsf{Z}^r(X \times C)$ each of whose components dominates C and points $t_1, t_2 \in C$ such that Z is of the form $W(t_1) - W(t_1)$, where

$$W(t_1) = \tau_* \circ \pi^*(t_1) \text{ and } W(t_2) = \tau_* \circ \pi^*(t_2)$$

with τ (resp. π) the restriction to W of the projection to X (resp. to C), see Remark 1.2.10.

Now consider the following commutative diagram

$$\begin{array}{ccc}
\mathsf{Z}^r(X) & \xrightarrow{cl} & H^{2r}(X) \\
\tau_* \uparrow & & \tau_* \uparrow \\
\mathsf{Z}^1(W) & \xrightarrow{cl} & H^2(W) \\
\pi^* \uparrow & & \pi^* \uparrow \\
\mathsf{Z}^1(C) & \xrightarrow{cl} & H^2(C)
\end{array}$$

Note that since C is connected the cycle map $cl : \mathsf{Z}^1(C) \to H^2(C) \simeq \mathbb{Z}$ coincides with the degree map $\deg : \mathsf{Z}_0(C) \to \mathbb{Z}$ of 0-cycles on C, so for $t_1, t_2 \in C$ we have $cl(t_1) = cl(t_2) = 1$, then

$$\tau_* \circ \pi^* \circ cl(t_1) = \tau_* \circ \pi^* \circ cl(t_2).$$

By the commutativity of the diagram we have

$$cl \circ \tau_* \circ \pi^*(t_1) = \tau_* \circ \pi^* \circ cl(t_1)$$

and similarly

$$\tau_* \circ \pi^* \circ cl(t_2) = cl \circ \tau_* \circ \pi^*(t_2),$$

it follows that

$$cl \circ \tau_* \circ \pi^*(t_1) = cl \circ \tau_* \circ \pi^*(t_2)$$

which is equivalent to

$$cl(W(t_1)) = cl(W(t_2)).$$

Since the cycle map is an homomorphism we have

$$cl(W(t_1) - W(t_2)) = cl(Z) = 0,$$

then $Z \sim_{\text{hom}} 0$, that is, $Z \in \mathsf{Z}^r(X)_{\text{hom}}$. □

The above proposition is also true over an arbitrary field, see [24, §1.2.1.].

Rational, algebraic and homological equivalence of 0-cycles

Let X be a smooth projective reduced scheme of dimension n over an algebraically closed field k of characteristic 0.

From the Proposition 1.2.18 for the case $r = n$, one has

$$\mathsf{Z}^n(X)_{\mathsf{alg}} \subset \mathsf{Z}^n(X)_{\text{hom}}$$

or equivalently

$$\mathsf{Z}_0(X)_{\mathsf{alg}} \subset \mathsf{Z}_0(X)_{\text{hom}}.$$

Now we prove that if in addition X is connected the other inclusion also holds, therefore $\mathsf{Z}_0(X)_{\text{hom}} = \mathsf{Z}_0(X)_{\mathsf{alg}}$. To prove it we use the following lemmas

Lemma 1.2.19. *(0-cycles homologous to 0 have degree 0 and vice versa) Assume in addition that X is connected, and let $\mathsf{Z}_0(X)_{\text{deg}=0} \subset \mathsf{Z}_0(X)$ be the group of 0-cycles of degree 0. Then*

$$\mathsf{Z}_0(X)_{\text{hom}} = \mathsf{Z}_0(X)_{\text{deg}=0}.$$

Proof. Since X is connected we have $H^{2n}(X) \simeq \mathbb{Z}$, i.e., we have

$$\begin{tikzcd} \mathsf{Z}_0(X) \arrow[r, "cl"] \arrow[dr, "\deg"'] & H^{2n}(X) \arrow[d, "\simeq"] \\ & \mathbb{Z} \end{tikzcd}$$

So,

$$\mathsf{Z}_0(X)_{\text{hom}} = \mathsf{Ker}(cl : \mathsf{Z}_0(X) \to H^{2n}(X)) = \mathsf{Ker}(\deg : \mathsf{Z}_0(X) \to \mathbb{Z}) = \mathsf{Z}_0(X)_{\deg=0},$$

i.e. the 0-cycles homologous to 0 coincide with the 0 cycles of degree 0. ☐

Lemma 1.2.20. *Let C be an integral (reduced and irreducible) smooth curve and $P_1, P_2 \in C$ two points in C, then*

$$P_1 \sim_{\mathsf{alg}} P_2.$$

Proof. We must show that there exists a smooth curve D, $W \in \mathsf{Z}^1(C \times D)$ i.e. a family of 0-cycles (or equivalently cycles of codimension 1) on C each of whose components dominates D, and points $t_1, t_2 \in D$ such that $W(t_1) = P_1$ and $W(t_2) = P_2$, i.e., such that P_1 and P_2 are members of this family.

It is enough to take $D = C$, $W = \Delta = \{(a, b) \in C \times C : a = b\} \subset C \times C$ the diagonal, and $t_1 = P_1$ and $t_1 = P_2$. ☐

Lemma 1.2.21. *If there exists a connected curve C such that its components are smooth and integral and P and Q are two points of C, then $P \sim_{\mathsf{alg}} Q$.*

Proof. Assume that such a curve $C = C_1 \cup \cdots \cup C_r$ exists, then without lost of generality we can assume that $P = P_0 \in C_1$, $Q = P_r \in C_r$ and that $P_i \in C_i \cap C_{i+1}$ for all $i = 1, \ldots, r-1$, then for the Lemma 1.2.20 we have $P_{i-1} \sim_{\mathsf{alg}} P_i$ for all $i = 1, \ldots, r$, then $P = P_0 \sim_{\mathsf{alg}} Q = P_r$. ☐

Proposition 1.2.22. *(Homological and algebraic equivalence coincide for 0-cycles) If X is connected, then*

$$\mathsf{Z}_0(X)_{\text{hom}} = \mathsf{Z}_0(X)_{\mathsf{alg}}.$$

Proof. By Proposition 1.2.18 it is enough to prove that $\mathsf{Z}_0(X)_{\text{hom}} \subset \mathsf{Z}_0(X)_{\mathsf{alg}}$. By Lemma 1.2.19 this is equivalent to prove that if a 0-cycle has degree zero then it is algebraically equivalent to 0, which we do next.

Let $Z = \sum_i m_i P_i$ be a 0-cycle on X with degree zero, i.e., with $\sum_i m_i = 0$, then we can consider $Z = P - Q$ with $P, Q \in X$, this holds true because since Z has degree zero it is generated by differences of this form. Indeed, observe that

$$Z = \sum m_i P_i = \sum_i m_i (P_i - Q) = \sum_i m_i P_i - \sum_i m_i Q,$$

for any $Q \in X$.

By [30, §8.2.1] there exits a smooth connected curve C containing $P, Q \in X$, then by Lemma 1.2.21 $P \sim_{\mathsf{alg}} Q$ or equivalently $Z = P - Q \sim_{\mathsf{alg}} 0$. $\square$

Finally, we next study the relation between rational, algebraic and homological equivalence of 0-Cycles on a smooth projective variety.

Proposition 1.2.23. *Let X be a smooth projective variety of dimension n over an algebraically closed field k of characteristic 0. Then*

$$\mathsf{CH}_0(X)_{\deg=0} = \mathsf{CH}_0(X)_{\hom} = \mathsf{A}_0(X).$$

Proof. Since by Lemma 1.2.19 we have $\mathsf{Z}_0(X)_{\deg=0} = \mathsf{Z}_0(X)_{\hom}$ the first equality holds.

On the other hand, by Proposition 1.2.22 we have that $\mathsf{Z}_0(X)_{\hom} = \mathsf{Z}_0(X)_{\mathsf{alg}}$, this gives the second equality. $\square$

If $\dim(X) = 1$, that is, X is a smooth projective curve over an algebraically closed field k of characteristic 0 we get the following important lemma for the proof of the main result of this thesis.

Lemma 1.2.24. *(Fact 2) Let C be a smooth projective curve over an algebraically closed field k of characteristic 0. Then*

$$\mathsf{CH}_0(C)_{\deg=0} = \mathsf{CH}_0(C)_{\hom} = \mathsf{A}_0(C).$$

Proof. Apply Proposition 1.2.23 when $\dim(X) = 1$. $\square$

1.3 Chow groups

Let X be a scheme. Recall that Chow group of r-cycles is defined as the group quotient

$$\mathsf{CH}_r(X) = \frac{\mathsf{Z}_r(X)}{\mathsf{Z}_r(X)_{\mathsf{rat}}},$$

see Definition 1.2.7.

We now study some properties of the Chow groups.

Theorem 1.3.1 (Rational equivalence pushes forward)**.** *If $f : X \to Y$ is a proper morphism and Z is a r-cycle on X rationally equivalent to zero, then $f_*(Z)$ is a r-cycle rationally equivalent to zero on Y.*

Proof. See [10, Theorem 1.4]. $\square$

By Theorem 1.3.1, given a proper morphism $f : X \to Y$ there is an induced homomorphisms on Chow groups

$$f_* : \mathsf{CH}_r(X) \longrightarrow \mathsf{CH}_r(Y).$$

Definition 1.3.2. The morphism f_* is called the *pushforward homomorphism* or *Gysin homomorphism* on Chow groups induced by f.

So, we see that the Chow groups have 'homological-like' properties.

On the other hand, consider the following lemma

Lemma 1.3.3. *Let $f : X \to Y$ be a flat morphism of relative dimension n and Z an a r-cycle on Y which is rationally equivalent to zero. Then $f^*(Z)$ is rationally equivalent to zero in $\mathsf{Z}_{r+n}(X)$.*

Proof. See [10, Theorem 1.7]. □

By Lemma 1.3.3 there is an induced homomorphism on Chow groups

$$f^* : \mathsf{CH}_r(Y) \to \mathsf{CH}_{r+n}(X).$$

Definition 1.3.4. The morphism f^* is called *pullback homomorphism* on Chow groups induced by f.

So, we see that the Chow groups have also 'cohomological-like' properties.

The group $\mathsf{CH}_0(X)_{\deg=0}$

Let X be a complete scheme over a field k, that is, X is proper over $\mathsf{Spec}(k)$.

Definition 1.3.5 (Degree of a 0-cycle)**.** The *degree of* 0-*cycles* on X is a homomorphism

$$\deg : \mathsf{Z}_0(X) \to \mathbb{Z},$$

defined by

$$Z = \sum_i n_i P_i \mapsto \deg(Z) = \sum_i n_i [k(P_i) : k],$$

where $k(P_i)$ denotes the residue field of the point P_i.

Claim: $\deg = f_*$, where $f : X \to \mathsf{Spec}(k)$ is the structural morphism.

Proof. Since X is complete, the structure morphism $f : X \to \mathsf{Spec}(k)$ is proper, so there is an induced homomorphism

$$f_* : \mathsf{Z}_0(X) \to \mathsf{Z}_0(\mathsf{Spec}(k))$$

defined by

$$f_*(Z) = \sum_i n_i f_*(P_i) = \sum_i n_i \deg(P_i/f(P_i)) f(P_i).$$

Since $\dim(P_i) = \dim(f(P_i)) = \dim(\mathsf{Spec}(k))$, we have

$$\deg(P_i/f(P_i)) = [k(P_i) : k(\mathsf{Spec}(k))] = [k(P_i) : k],$$

then

$$f_*(Z) = \sum_i n_i [k(P_i) : k] \mathsf{Spec}(k).$$

Sending $\mathsf{Spec}(k) \to 1$, we get $\deg = f_*$. $\square$

By Theorem 1.3.1 we have an induced homomorphism, which we also denote by deg, on Chow groups

$$\deg : \mathsf{CH}_0(X) \to \mathsf{CH}_0(\mathsf{Spec}(k)).$$

Since $Z_0(\mathsf{Spec}(k)) = \mathsf{CH}_0(\mathsf{Spec}(k)) = \mathbb{Z}$ we get

$$\deg : \mathsf{CH}_0(X) \to \mathbb{Z}$$

the degree homomorphism on the Chow group of 0-cycles.

Definition 1.3.6 (The group $\mathsf{CH}_0(X)_{\deg=0}$)**.** The kernel of $\deg : \mathsf{CH}_0(X) \to \mathbb{Z}$ is denoted by

$$\mathsf{CH}_0(X)_{\deg=0} = \mathsf{Ker}(\deg : \mathsf{CH}_0(X) \to \mathbb{Z}).$$

This is the *Chow group of* 0*-cycles of degree zero* on X.

In particular, it follows that rationally equivalent cycles have the same degree.

Countability lemmas

In this subsection we present some countability lemmas needed to prove the main result of the thesis.

Let k be an uncountable field. In this subsection a variety is a reduced scheme, not necessarily irreducible.

Lemma 1.3.7. *Let X be an irreducible quasi-projective algebraic variety over k. Then X can not be written as a countable union of its Zariski closed subsets, each of which is not the whole X.*

Proof. See [1, Lemma 10]. $\square$

Definition 1.3.8 (Irredundant countable union)**.** A countable union $V = \cup_{n\in\mathbb{N}} V_n$ of algebraic varieties will be called *irredundant* if V_n is irreducible for each n and $V_m \not\subset V_n$ for $m \neq n$. If V is a irredundant decomposition, then the sets V_n are called c-components of V.

Lemma 1.3.9. *Let V be a countable union of algebraic varieties over an uncountable algebraically closed ground field. Then V admits an irredundant decomposition, and such an irredundant decomposition is unique.*

Proof. See [1, Lemma 11]. $\square$

Lemma 1.3.10. *Let A be an abelian variety over k, and let K be a subgroup which can be represented as a countable union of Zariski closed subsets in A. Then the irredundant decomposition of K contains a unique irreducible component passing through 0, and this component is an Abelian subvariety in A.*

Proof. See [1, Lemma 12]. □

Regular maps into $\mathsf{CH}_0(X)$

Let X be a nonsingular projective variety over an uncountable algebraically closed field of characteristic zero.

Definition 1.3.11 (c-closed, c-open)**.** A subset of an integral scheme T which is union of a countable number of closed subsets is called a *c-closed* subset and the complement of a c-closed, i.e., intersections of a countable number of open subsets is called a *c-open* subset.

Definition 1.3.12 (Very general notion)**.** Let T be an integral scheme. We say that a *property Q holds for a very general point on T* if there exists a c-open subset in T such that Q holds for each point in this c-open.

Here we work over an uncountable field because in this case the theorem on unique decomposition into irreducible components extends to c-closed subsets, so we can speak about the dimension of a c-closed subset, understanding by this the maximum of the dimensions of its irreducible components.

Definition 1.3.13 (Symmetric product)**.** The *d-th symmetric product* of a variety X, denoted by $\mathsf{Sym}^d(X)$, is the quotient variety $\mathsf{Sym}^d(X) = X^d/\Sigma_d$, where X^d is the self-product of X and Σ_d is the group of permutations of the factors.

The d-th symmetric product $\mathsf{Sym}^d(X)$ is a variety of dimension nd, where $n = \dim(X)$ and as a set coincides with the set of effective 0-cycles of degree d, i.e.,

$$\mathsf{Sym}^d(X) \underset{\text{as set}}{=} \{ \text{ effective 0-cycles of degree } d \text{ on } X\}.$$

Definition 1.3.14 (Difference map)**.** The set-theoretic map

$$\begin{array}{rccl} \theta^X_{d_1,d_2}: & \mathsf{Sym}^{d_1}(X) \times \mathsf{Sym}^{d_2}(X) & \to & \mathsf{CH}_0(X) \\ & (A,B) & \mapsto & [A-B] \end{array}$$

where $[A-B]$ is the class of the cycle $A-B$ modulo rational equivalence, will be called the *difference map.*

Remark 1.3.15. When $d_1 = d_2 = d$ we will denote $\theta^X_{d_1,d_2}$ just by θ^X_d.

For any non negative integers $d_1, \ldots, d_s$ we denote by

$$\mathsf{Sym}^{d_1,\ldots,d_s}(X) = \mathsf{Sym}^{d_1}(X) \times \cdots \times \mathsf{Sym}^{d_s}(X).$$

to the fibred product over the ground field k.

Let

$$\begin{aligned} W^{d_1,d_2} &= \{(A,B;C,D) \in \mathsf{Sym}^{d_1,d_2}(X) \times \mathsf{Sym}^{d_1,d_2}(X) : \theta^X_{d_1,d_2}(A,B) = \theta^X_{d_1,d_2}(C,D)\} \\ &= \{(A,B;C,D) \in \mathsf{Sym}^{d_1,d_2}(X) \times \mathsf{Sym}^{d_1,d_2}(X) : (A-B) \sim_{\mathsf{rat}} (C-D)\} \end{aligned}$$

be the subset of $\mathsf{Sym}^{d_1,d_2}(X) \times \mathsf{Sym}^{d_1,d_2}(X)$ defining the rational equivalence on $\mathsf{Sym}^{d_1,d_2}(X)$. It is a c-closed subset by Lemma 1 in [27].

Remark 1.3.16. Note that W^{d_1,d_2} is the fibred product

$$\begin{tikzcd} W^{d_1,d_2} = \mathsf{Sym}^{d_1,d_2}(X) \times_{\mathsf{CH}_0(X)} \mathsf{Sym}^{d_1,d_2}(X) \arrow[r] \arrow[d] & \mathsf{Sym}^{d_1,d_2}(X) \arrow[d, "\theta^X_{d_1,d_2}"] \\ \mathsf{Sym}^{d_1,d_2}(X) \arrow[r, "\theta^X_{d_1,d_2}"] & \mathsf{CH}_0(X) \end{tikzcd}$$

Definition 1.3.17 (Regular map into $\mathsf{CH}_0(X)$)**.** A set-theoretic map $\kappa : Z \to \mathsf{CH}_0(X)$ of an algebraic variety Z into the Chow group of 0-cycles $\mathsf{CH}_0(X)$ will be called *regular* if there exists a commutative diagram (in the set-theoretic sense)

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "g"] & \mathsf{Sym}^{n,m}(X) \arrow[d, "\theta^X_{d_1,d_2}"] \\ Z \arrow[r, "\kappa"] & \mathsf{CH}_0(X) \end{tikzcd}$$

where f is a regular map and g is an epimorphism which is also a regular map.

Definition 1.3.18 (Algebraic subset of $\mathsf{CH}_0(X)$)**.** The image $\kappa(Z) \subset \mathsf{CH}_0(X)$ of a regular map $\kappa : Z \to \mathsf{CH}_0(X)$ will be called an *algebraic subset* of $\mathsf{CH}_0(X)$.

Definition 1.3.19 (Closed subset of $\mathsf{CH}_0(X)$)**.** If Z is projective (irreducible), then its image $\kappa(Z) \subset \mathsf{CH}_0(X)$ under a regular map $\kappa : Z \to \mathsf{CH}_0(X)$ will be called *closed subset (irreducible)* of $\mathsf{CH}_0(X)$.

Equivalently set-theoretic regular maps into $\mathsf{CH}_0(X)$ can be defined as follows ([27, Lemma 4]).

Definition 1.3.20 (Alternative definition of a regular map into $\mathsf{CH}_0(X)$)**.** The set-theoretic map $\kappa : Z \to \mathsf{CH}_0(X)$ is *regular* if and only if for any integers d_1 and d_2 the subset

$$W_{\kappa,\theta^X_{d_1,d_2}} = \{(z,A,B) \in Z \times \mathsf{Sym}^{d_1,d_2}(X) : \kappa(z) = \theta^X_{d_1,d_2}(A,B)\} = Z \times_{\mathsf{CH}_0(X)} \mathsf{Sym}^{n,m}(X)$$

is c-closed.

Lemma 1.3.21. *The map* $\theta^X_{d_1,d_2} : \mathsf{Sym}^{d_1,d_2}(X) \to \mathsf{CH}_0(X)$ *is regular.*

Proof. It follows from the above alternative definition of a regular map into $\mathsf{CH}_0(X)$ and the fact that the subset

$$W^{d_1,d_2} = \mathsf{Sym}^{d_1,d_2}(X) \times_{\mathsf{CH}_0(X)} \mathsf{Sym}^{d_1,d_2}(X)$$

is c-closed [27, Lemma 1]. □

Remark 1.3.22. Recall that $\mathsf{CH}_0(X) = \mathbb{Z} \times \mathsf{CH}_0(X)_{\deg=0}$, where $\mathsf{CH}_0(X)_{\deg=0}$ is the Chow group of 0-cycles of degree zero, see [27, Introduction].

Lemma 1.3.23. *Let* $\kappa : Z \to \mathsf{CH}_0(X)_{\deg=0}$ *be a regular map and let*

$$alb_X : \mathsf{CH}_0(X)_{\deg=0} \to \mathrm{Alb}(X)$$

be the Albanese map. Then the composite map $alb_X \circ \kappa : Z \to \mathrm{Alb}(X)$ *is a regular map of algebraic varieties.*

Proof. see [27, Lemma 8]. □

Representability of $\mathsf{CH}_0(X)_{\deg=0}$

Let X be a connected smooth projective variety over $\mathbb{C}$ of dimension n.

Definition 1.3.24 (Representability)**.** $\mathsf{CH}_0(X)_{\deg=0}$ is *representable* if the natural map

$$\theta^X_d : \mathsf{Sym}^d(X) \times \mathsf{Sym}^d(X) \to \mathsf{CH}_0(X)_{\deg=0}$$

is surjective for sufficiently large d (see [30, Definition 10.6]).

Equivalently the representability of $\mathsf{CH}_0(X)_{\deg=0}$ can be defined as follows ([30, Theorem 10.11]).

Definition 1.3.25 (Representability)**.** $\mathsf{CH}_0(X)_{\deg=0}$ is *representable* if and only if

$$alb_X : \mathsf{CH}_0(X)_{\deg=0} \to \mathrm{Alb}(X)$$

is an isomorphism.

The following Lemma is a basic property of rational equivalence ([30, Lemma 10.7], [22, Lemma 3]).

Lemma 1.3.26. *The fibres of*

$$\theta^X_d : \mathsf{Sym}^d(X) \times \mathsf{Sym}^d(X) \to \mathsf{CH}_0(X)_{\deg=0}$$

are countable unions of closed algebraic subsets of $\mathsf{Sym}^d(X) \times \mathsf{Sym}^d(X)$*.*

If we denote by $(\theta_d^X)^{-1}(\alpha)$, $\alpha \in \mathsf{CH}_0(X)_{\deg=0}$, to any fiber of the set-theoretic map θ_d^X, the above lemma tell us

$$(\theta_d^X)^{-1}(\alpha) = \bigcup_{countable} \text{closed algebraic subsets of } \mathsf{Sym}^d(X) \times \mathsf{Sym}^d(X)$$

So now we can talk about the dimension of a fiber of θ_d^X.

Definition 1.3.27 (Dimension of any fibre of θ_d^X)**.** For any $\alpha \in \mathsf{CH}_0(X)_{\deg=0}$, we have

$$\dim(\theta_d^X)^{-1}(\alpha) = \text{largest dimension among the dimensions of its algebraic components}$$

Property: There is a countable union B of proper algebraic subsets of $\mathsf{Sym}^d(X) \times \mathsf{Sym}^d(X)$ such that for $x \in \mathsf{Sym}^d(X) \times \mathsf{Sym}^d(X) \setminus B$,

$$\dim((\sigma_d^X)^{-1}(\sigma_d^X(x))) = r_d,$$

is constant and equal to r_d (see [30, page 282]).

Definition 1.3.28 (Dimension of a very general fibre of θ_d^X)**.** r_d is called the *dimension of a very general fiber* of θ_d^X.

Remark 1.3.29. In [30], for the above definition "general" is used instead of "very general".

Then the dimension of a closed subset of $\mathsf{CH}_0(X)$, that is, the dimension of the image of the regular map θ_d^X is defined as follows

Definition 1.3.30 (Dimension of a closed subset of $\mathsf{CH}_0(X)$)**.** The *dimension of the image* of the map θ_d^X is

$$\dim(\mathrm{im}(\theta_d^X)) \quad = \quad \dim(\mathsf{Sym}^d(X) \times \mathsf{Sym}^d(X)) - r_d = 2nd - r_d.$$

Definition 1.3.31 (Dimension of $\mathsf{CH}_0(X)_{\deg=0}$)**.** We say that $\mathsf{CH}_0(X)_{\deg=0}$ is *infinite dimensional* if

$$lim_{d\to\infty} \dim \mathrm{im}(\sigma_d^X) = \infty,$$

and *finite dimensional* otherwise.

Then the representability of $\mathsf{CH}_0(X)_{\deg=0}$ can be defined as follows ([30, Proposition 10.10]).

Proposition 1.3.32. *The group $\mathsf{CH}_0(X)_{\deg=0}$ is representable if and only if it is finite dimensional.*

1.4 Divisors and Linear system

Weil divisors

Let X be a variety of dimension n over an algebraically closed field.

Definition 1.4.1 (Weil divisors)**.** A *Weil divisor* on X is a finite formal sum

$$D = \sum_i n_i D_i,$$

where $n_i \in \mathbb{Z}$ and D_i are subvarieties of dimension $n-1$, i.e., of codimension 1 on X. In other words, a Weil divisor on X is a cycle of codimension 1 on X.

Example 1.4.2. A Weil divisor on a surface is a finite formal sum $D = \sum_i n_i D_i$ where $n_i \in \mathbb{Z}$ and D_i are curves (i.e. subvarieties of dimension 1).

The set of divisors on X is denoted by $\mathsf{Div}(X)$. It is a group with the sum of divisors defined by

$$\sum_i n_i D_i + \sum_i m_i D_i = \sum_i (n_i + m_i) D_i.$$

Definition 1.4.3 (Effective Weil divisor)**.** A Weil divisor is called *effective* if $n_i \geq 0$ for all n_i.

Let $f \in k(X)^*$ be a rational function on X, let $div(f)$ be the $(n-1)$-cycle or divisor associated to f (see Definition 1.2.1).

Definition 1.4.4 (Principal divisors)**.** Any divisor which is equal to the divisor of a rational function is called a *principal divisor*.

Definition 1.4.5 (Locally principal Weil divisors)**.** A Weil divisor D is *locally principal* if X can be covered by open sets U such that $D\mid_U$ is principal.

Definition 1.4.6 (Linear equivalence of Weil divisors)**.** Two divisors D_1 and D_2 are said to be *linearly equivalent*, denoted by $D_1 \sim_{\mathsf{lin}} D_2$, if $D_1 - D_2$ is a principal divisor.

Cartier divisors

In this subsection let X be any arbitrary scheme.

Let $\mathscr{K}$ be the sheaf of total quotient rings of $\mathcal{O}_X$ ([15, §6]), let $\mathscr{K}^*$ be the sheaf of invertible elements in the sheaf of rings $\mathscr{K}$, and let $\mathcal{O}_X^*$ be the sheaf of invertible elements in the sheaf $\mathcal{O}_X$.

Remark 1.4.7. The sheaf $\mathscr{K}$ is a generalization of the concept of function field of an integral scheme.

Definition 1.4.8 (Cartier divisor)**.** A *Cartier divisor* on X is a global section of $\frac{\mathscr{K}^*}{\mathcal{O}_X^*}$.

A Cartier divisor on X can be described by the data $\{(U_i, f_i)\}$, where U_i form an open cover of X and f_i is an element of $\Gamma(U_i, \mathscr{K}^*) = \mathscr{K}^*(U_i)$ such that $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$ for each i, j.

Definition 1.4.9 (Principal Cartier divisor)**.** A Cartier divisor is *principal* if it is in $\mathrm{im}(\Gamma(X, \mathscr{K}^*) \to \Gamma(X, \frac{\mathscr{K}^*}{\mathcal{O}_X^*}))$.

Definition 1.4.10 (Linear equivalence of Cartier divisors)**.** Two Cartier divisors are *linearly equivalent* if their difference is principal.

Remark 1.4.11. When X is a nonsingular variety the group $\mathsf{Div}(X)$ of Weil divisors on X is isomorphic to the group $\Gamma(X, \frac{\mathscr{K}^*}{\mathcal{O}_X^*})$ of Cartier divisors on X and principal Weil divisors correspond to principal Cartier divisors ([15, Remark 6.11.1A.]). So, in this case usually one refers to them as divisors without any qualifier. In general, the group of Cartier divisors coincides with the group of locally principal Weil divisors.

Definition 1.4.12 (Effective Cartier divisor)**.** A Cartier divisor on X is *effective* if it can be represented by $\{(U_i, f_i)\}$ where $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$.

Definition 1.4.13 (Associated subscheme of an effective Cartier divisor)**.** Let D be an effective Cartier divisor represented by $\{(U_i, f_i)\}$. The *associated subscheme of codimension* 1 is the closed subscheme defined by the sheaf of ideals $\mathscr{I}$ which is locally generated by f_i.

The above definition gives a one to one correspondence between effective Cartier divisors and closed subschemes that are locally principal (i.e., subschemes whose sheaf of ideals are locally generated by a single element).

Remark 1.4.14. If X is a nonsingular variety (so, Cartier divisors coincide with Weil divisors) we have a one to one correspondence between effective Weil divisors and closed subschemes that are locally principal.

Let us now see that there is a one to one correspondence between Cartier divisors and some special invertible sheaves.

Definition 1.4.15 (Invertible sheaf)**.** An invertible sheaf on a ringed space is a locally free sheaf of rank 1.

Definition 1.4.16 (Picard group)**.** The group of invertible sheaves on a ringed space X modulo isomorphism is called the *Picard group* of X and is denoted by $\mathrm{Pic}(X)$.

Definition 1.4.17 (Invertible sheaf associated to a Cartier divisor)**.** Let D be a Cartier divisor on X represented by $\{(U_i, f_i)\}$ as above. The sheaf $\mathcal{O}_X(D)$ associated to D is a subsheaf of $\mathscr{K}$ defined by taking $\mathcal{O}_X(D)$ to be the sub $\mathcal{O}_X$-module of $\mathscr{K}$ generated by f_i^{-1} on U_i.

It is known that $\mathcal{O}_X(D)$ is an invertible sheaf on X and that the map $D \mapsto \mathcal{O}_X(D)$ gives a one to one correspondence between Cartier divisors on X and invertible sub sheaves of $\mathscr{K}$ (see [15, Proposition 6.13]).

Remark 1.4.18. In a projective or integral scheme every invertible sheaf is isomorphic to a subsheaf of $\mathscr{K}$. So, when X is projective and nonsingular variety (so, Cartier divisors coincide with Weil divisors), we have that

$$\mathsf{CH}^1(X) \to \mathrm{Pic}(X)$$

is an isomorphism.

In the rest of this section we will see that to give a morphism of a scheme X to a projective space is equivalent to give an invertible sheaf and a set of its global section generating it ([15, Theorem 7.1]).

Let R be a fixed ring. Let $\mathbb{P}^n_R = \mathrm{Proj}(R[x_0, \dots, x_n])$ be the projective space over R. Let X be any scheme over R. Recall that on $\mathbb{P}^n_R$ there is an invertible sheaf $\mathcal{O}_{\mathbb{P}^n_R}(1)$ which is generated by the global sections $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(1))$ induced by the homogeneous coordinates $x_0, \dots, x_n$.

If $\varphi : X \to \mathbb{P}^n_R$ is an R-morphism of X to $\mathbb{P}^n_R$, then $\varphi^*(\mathcal{O}_{\mathbb{P}^n_R}(1))$ is an invertible sheaf on X and the global sections $s_i = \varphi^*(x_i) \in \Gamma(X, \varphi^*(\mathcal{O}_{\mathbb{P}^n_R}(1)))$, $i = 0, \dots, n$, generate the sheaf $\varphi^*(\mathcal{O}_{\mathbb{P}^n_R}(1))$.

Conversely, if $\mathscr{L}$ is an invertible sheaf on X, and if $s_i \in \Gamma(X, \mathscr{L})$, $i = 0, \dots, n$ are global sections generating $\mathscr{L}$, then there exits a unique R-morphism $\varphi : X \to \mathbb{P}^n_R$ such that $\mathscr{L} \cong \varphi^*(\mathcal{O}_{\mathbb{P}^n_R}(1))$ and $s_i = \varphi^*(x_i)$ under this isomorphism

To give some criteria for a morphism induced by an invertible sheaf to be an immersion we need the following definition

Definition 1.4.19 (Very ample invertible sheaf)**.** Let X be any scheme over Y. An invertible sheaf $\mathscr{L}$ on X is *very ample relative to* Y if there exists an immersion $i : X \to \mathbb{P}^r_Y$ for some r, such that $\mathscr{L} \simeq i^*(\mathcal{O}_{\mathbb{P}^r_Y}(1))$.

When $Y = \mathsf{Spec}(k)$, where k is a field, we have the following criteria to determine when the morphism to the projective space induced by an invertible sheaf is an immersion.

Proposition 1.4.20. *Let X be a scheme over $\mathsf{Spec}(k)$. An invertible sheaf $\mathscr{L}$ is very ample (relative to $\mathsf{Spec}(k)$) if $\mathscr{L}$ admits a set of global sections s_i, $i = 0, \dots, n$, such that the corresponding morphism $\varphi : X \to \mathbb{P}^n_k$ is an immersion (see [15, §7]).*

Definition 1.4.21 (Very ample divisor)**.** A divisor D on a nonsingular and projective variety X is called *very ample* if $\mathcal{O}_X(D)$ is very ample.

Linear System

Let X be a nonsingular projective variety over an algebraically closed field k (note that in this case the notion of Cartier divisors is equivalent to the notion of Weil divisors; any invertible sheaf is isomorphic to a subsheaf of $\mathscr{K}$; there is a one to one correspondence between linear equivalence classes of divisors and isomorphism classes of invertible sheaves; and for any invertible sheaf $\mathscr{L}$ on X, $\Gamma(X, \mathscr{L})$ is a finite dimensional vector space).

The following definition shows that global sections of an invertible sheaf on X correspond to effective divisors on X.

Definition 1.4.22 (Divisor of zeros of a section)**.** Let $\mathscr{L}$ be an invertible sheaf on X, and let $s \in \Gamma(X, \mathscr{L})$ be a nonzero section of $\mathscr{L}$. The *divisor of zeros of s* is a (Cartier) divisor, denoted by $(s)_0$, described by the data $\{U_i, \varphi_{U_i}(s)\}$, where U_i is any open set of X in which $\mathscr{L}$ is trivial and $\varphi_{U_i} : \mathscr{L} \mid_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$ is an isomorphism. Note in particular that as U_i ranges in a covering of X, the collection $\{U_i, \varphi_{U_i}(s)\}$ determines an effective Cartier divisor $(s)_0$.

Proposition 1.4.23. *Let D be a divisor on X and let $\mathscr{L} \cong \mathcal{O}_X(D)$ be the corresponding invertible sheaf. Then*

a) For each nonzero $s \in \Gamma(X, \mathscr{L})$ we have that $(s)_0 \sim_{\mathsf{lin}} D$ and is effective.

b) Every effective divisor D' linearly equivalent D is the divisor of zeros of a section of $\mathscr{L}$, i.e., $D' = (s)_0$ for some $s \in \Gamma(X, \mathscr{L})$.

c) Two sections $s, s' \in \Gamma(X, \mathscr{L})$ have the same divisor of zeros if and only if $s = ts'$ for some $t \in k^*$.

Proof. See [15, Proposition 7.7.]. □

Definition 1.4.24 (Complete linear system)**.** A *complete linear system* on a nonsingular projective variety is the set of all effective divisors linearly equivalent to some given divisor D, usually denoted by $\mid D \mid$.

It follows from Proposition 1.4.23 that the complete linear system $\mid D \mid$ is in one to one correspondence with the set $\frac{\Gamma(X,\mathscr{L})-\{0\}}{k^*} = \mathbb{P}(\Gamma(X, \mathscr{L}))$, that is, we can define the complete linear system as follows: $\mid D \mid = \mathbb{P}(\Gamma(X, \mathscr{L})) = \mathbb{P}(H^0(X, \mathscr{L}))$. So, $\mid D \mid$ has the structure of the set of closed points of a projective space over k and it is also sometimes denoted by $\mid \mathscr{L} \mid$.

Definition 1.4.25 (Linear system)**.** A linear system δ on X is a subset of a complete linear system $\mid D \mid$ which is a linear subspace for the projective space structure of $\mid D \mid$.

A linear system δ corresponds to a vector subspace

$$V = \{s \in \Gamma(X, \mathscr{L}(D)) : (s)_0 \in \delta\} \cup \{0\}.$$

Definition 1.4.26 (Dimension of a linear system)**.** The *dimension* of the linear system is defined by

$$\dim(\delta) = \dim(V) - 1.$$

Definition 1.4.27 (Support of a Divisor)**.** The *support* of a divisor $D = \sum_i n_i D_i$ is $\bigcup D_i$, the union of the subvarieties of codimension 1, it is denoted by $\mathsf{Supp}(D)$.

Definition 1.4.28 (Base point)**.** A point $x \in X$ is a *base point* of a linear system δ if $x \in \mathsf{Supp}(D)$ for all $D \in \delta$.

Definition 1.4.29. (Fixed part of a linear system) A fixed part of a of a linear system δ is the biggest divisor F that is contained in every element of δ.

Lemma 1.4.30. *Let δ be a linear system on X corresponding to the subspace $V \subset \Gamma(X, \mathscr{L})$. Then δ is base point free if and only if $\mathscr{L}$ is generated by global sections in V.*

Proof. See [15, Lema 7.8.] □

So giving a morphism $X \to \mathbb{P}^n_k$ is equivalent to giving a linear system δ without base points on X and a set of elements $s_0, \ldots, s_n \in V$ which span V.

Lemma 1.4.31. *Let S be a smooth projective surface over the complex numbers, let D a very ample divisor on S and $\mathscr{L} = \mathcal{O}_S(D)$ its corresponding very ample invertible sheaf on S and let $\Sigma =\mid D \mid=\mid \mathscr{L} \mid$ the complete linear system corresponding to D on S of dimension say d and $\varphi_{\mathscr{L}} = \varphi_\Sigma : S \to \mathbb{P}^d$ the embedding corresponding to Σ, then S is nondegenerate, that is, is not contained in any hyperplane of $\mathbb{P}^d$.*

Proof. Since $\mathscr{L} = \mathcal{O}_S(D)$ is very ample and by Proposition 1.4.20 it admits a set of global sections $s_0, \ldots, s_d \in H^0(S, \mathcal{O}_S(D))$ generating it which by Lemma 1.4.30 this is equivalent to the fact that Σ is base points free then it has no fixed part, it follows by [3, II.6] that S is not degenerated. □

Remark 1.4.32. Let $\varphi : X \to \mathbb{P}^n_k$ be a morphism corresponding to the linear system δ. Then φ is a closed immersion if and only if δ separate points and separates tangent vectors (see [15, Remark 7.8.2]).

Chapter 2

Hodge Theory and The Abel-Jacobi Map

In this chapter we recall some facts about Hodge theory. We start with the notion of Hodge structure and polarized varieties which thanks to Kodaira Embedding Theorem are projective varieties, i.e., admits a holomorphic embedding into a projective space, then we study the notion of complex torus associated to the Hodge structure on the cohomology group $H^1(X)$ of a compact Kähler manifold X showing that it coincides with the Picard group $\mathrm{Pic}^0(X)$ and that if X is smooth and projective $\mathrm{Pic}^0(X)$ is an abelian variety. Next we define the morphism of Hodge structures showing that the pullback and the Gysin homomorphism on cohomology groups are two important examples of it.

In this chapter we also define the k-th intermediate Jacobian which is a complex torus associated to the $(2k-1)$-th cohomology group of X but which can be also defined for any Hodge structure of weight $2k-1$, then we see that the Jacobian torus $J(X)$ coincides the torus $\mathrm{Pic}^0(X)$ hence if X is smooth and projective the Jacobian has the structure of an abelian variety, we next study the relation between the cohomology group of a curve and its Jacobian. This topics gives us the background to prove *Fact 3* (Lemma 2.3.6). After that, we study the Abel Jacobi map which is a map from the group of k-cycles homologous to zero to the k-th intermediate Jacobian and which provide us with the necessary background to prove *Fact 1* (Lemma 2.4.12). Finally, we define the Albanese map and the Albanese variety.

2.1 Hodge structure and Polarized varieties

First recall the definition of a Kähler manifold.

Definition 2.1.1 (Kähler manifold)**.** A *Kähler manifold* is a complex manifold equipped with a hermitian metric whose imaginary part, which is a 2-form of type $(1,1)$ relative to the complex structure, is closed. This 2-form is called the *Kähler form*.

Example 2.1.2. Smooth projective complex manifolds are special cases of compact Kähler manifolds.

Hodge structure

Definition 2.1.3 (Integral Hodge structure of weight k)**.** An *integral Hodge structure of weight k* is given by a free abelian group $V_{\mathbb{Z}}$ of finite type, together with a decomposition (of its complexification)

$$V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$$

such that $V^{p,q} = \overline{V^{q,p}}$.

Definition 2.1.4 (Integral Hodge substructure)**.** An *integral Hodge substructure* is a subgroup $W_{\mathbb{Z}} \subset V_{\mathbb{Z}}$ such that $W_{\mathbb{C}} := W_{\mathbb{Z}} \otimes \mathbb{C}$ has a decomposition induced by that of $V_{\mathbb{C}}$, i.e., $W_{\mathbb{C}} = \bigoplus_{p+q=k} W_{\mathbb{C}} \cap V^{p,q}$.

Example 2.1.5. The integral cohomology group $H^k(X, \mathbb{Z})$ of a compact Kähler manifold carries a weight k Hodge structure. Indeed, recall that given a compact complex manifold X, there is an isomorphism

$$\mathcal{H}^k(X) \cong H^k(X, \mathbb{C}),$$

where $\mathcal{H}^k(X)$ is the set of complex valued harmonic forms for the Laplacian associated to any metric on X. When the metric is Kähler there is a decomposition of harmonic forms into harmonic forms of type (p, q). Thus, there is an induced decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X)$ is the set of classes representable by closed forms of type (p, q) and it satisfies the Hodge symmetry:

$$H^{p,q}(X) = \overline{H^{q,p}(X)}.$$

This decomposition is called the Hodge decomposition of the cohomology of a compact Kähler manifold ([29, §6.1.3]).

Given such a decomposition, we define the *associated Hodge filtration $F^\bullet V$* by

$$F^p V_{\mathbb{C}} = \bigoplus_{r \geq p} V^{r,k-r} = V^{p,k-p} \oplus \cdots \oplus V^{k,0}.$$

It is a decreasing filtration on $V_{\mathbb{C}}$, which satisfies

$$V_{\mathbb{C}} = F^p V_{\mathbb{C}} \oplus \overline{F^{k-p+1} V_{\mathbb{C}}}.$$

Remark 2.1.6. Hodge filtration determines the Hodge decomposition by

$$V^{p,q} = F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}.$$

Example 2.1.7. If X is a compact Kähler manifold and $V_{\mathbb{Z}} = H^k(X, \mathbb{Z})$, then

$$F^p H^k(X, \mathbb{C}) = \frac{\mathsf{Ker}(d : F^p A^k(X) \to F^p A^{k+1}(X))}{\mathrm{im}(d : F^p A^{k-1}(X) \to F^p A^k(X))},$$

where $F^p A^k(X)$ is the set of complex differential forms which are sums of forms of type $(r, k-r)$ with $r \geq p$ at every point.

In a wider context we have the following definition of a weight k Hodge structure ([26, §2.1.1]).

Definition 2.1.8. (A weight k Hodge structure) Let V be a finite dimensional real vector space and $V_{\mathbb{C}} = V \otimes \mathbb{C}$ its complexification.

- A *real Hodge structure* on V is a direct sum decomposition of its complexification, i.e.,

$$V_{\mathbb{C}} = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q} \text{ with } V^{p,q} = \overline{V^{q,p}}.$$

- If V has the real Hodge structure and is of the form $V = V_R \otimes_R \mathbb{R}$, where R is a sub-ring of $\mathbb{R}$ and V_R is an R-module of finite type, then we say that V_R carries and *R-Hodge structure*.

- If V is a real Hodge structure, the *weight k part* , denoted by $V^{(k)}$, is the real vector space underlying $\bigoplus_{p+q=k} V^{p,q}$.

- If $V = V^{(k)}$, we say that V is a *weight k real Hodge structure* and if $V = V_R \oplus_R \mathbb{R}$ we say that V_R is a *weight k R-Hodge structure*.

In what follows we will define a Polarized Hodge structure.

Let X be an n-dimensional compact Kähler manifold of Kähler form ω, then the cup product with the class $[\omega] \in H^2(X, \mathbb{R})$ of ω gives the Lefschetz operator

$$L : H^k(X, \mathbb{R}) \to H^{k+2}(X, \mathbb{R}).$$

This operator gives:

- the *Lefschetz decomposition*

$$H^k(X, \mathbb{R}) = \bigoplus_r L^r H^{k-2r}_{prim},$$

  where each component admits an induced Hodge decomposition,

- an *intersection form* on $H^k(X, \mathbb{R})$ for $k \leq n$

$$Q(\alpha, \beta) = \int_X \omega^{n-k} \wedge \alpha \wedge \beta = \langle L^{n-k}\alpha, \beta \rangle.$$

  Q is alternating if k is odd, symmetric otherwise.

Then we have an Hermitian form

$$H(\alpha, \beta) = i^k Q(\alpha, \overline{\beta})$$

on $H^k(X, \mathbb{C})$, induced by the intersection form Q,

Definition 2.1.9 (Integral Kähler form)**.** The class $[\omega]$ of the Kähler form ω of X is called *integral* if $[\omega]$ belongs to $H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R})$.

Definition 2.1.10 (Polarized Hodge structure)**.** An *integral polarised Hodge structure of weight* k is given by a Hodge structure $(V_{\mathbb{Z}}, F^p V_{\mathbb{C}})$ of weight k, together with an intersection form Q on $V_{\mathbb{Z}}$, which is symmetric if k is even, alternating otherwise, and satisfies conditions (i) and (ii) in [29, §7.1.2].

Polarized varieties

Definition 2.1.11 (Polarized manifold)**.** A *polarized manifold* is a pair $(X, [\omega])$, where X is a compact complex manifold, and $[\omega]$ is an integral Kähler class on X.

Definition 2.1.12 (Chern form)**.** Let X be a complex manifold, $\mathcal{L}$ a holomorphic line bundle on X, and h a Hermitian metric on $\mathcal{L}$. The 2-form $\omega_{\mathcal{L},h}$, which is closed and real of type $(1, 1)$, is called the *Chern form* associated to the hermitian metric h on $\mathcal{L}$. We say that $\omega_{\mathcal{L},h}$ is *positive* if it correspond to a Hermitian metric on X ([29, 3.3.1]).

As a consequence of Theorem 7.10 in [29], given a polarised manifold $(X, [\omega])$ there exits a holomorphic line bundle $\mathcal{L}$ on X and a Hermitian metric h such that $\omega_{\mathcal{L},h} = \omega$ is a positive form. We say that $\mathcal{L}$ is positive and we have the following theorem called Kodaira Embedding Theorem.

Theorem 2.1.13 (Kodaira Embedding Theorem)**.** *Let X be a compact complex manifold and $\mathcal{L}$ a positive holomorphic line bundle on X. Then for every sufficiently large $N \in \mathbb{Z}$ there exists a holomorphic embedding*

$$\phi : X \to \mathbb{P}^r$$

such that $\phi^*(\mathcal{O}_{\mathbb{P}^r}(1)) = \mathscr{L}^{\otimes N}$, where $\mathscr{L}$ is the sheaf of holomorphic sections of $\mathcal{L}$.

Proof. See [29, Theorem 7.11]. □

Corollary 2.1.14. *A polarized manifolds is a projective variety, i.e., admits a holomorphic embedding into a projective space.*

Proof. It follows from the comment above the Kodaira Embedding Theorem together with the Kodaira Embedding Theorem. □

Abelian variety associated to the Hodge structure of weight 1

Let X be compact Kähler manifold. The Hodge structure on $H^1(X)$ is described by the decomposition

$$H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$$

satisfying $H^{0,1}(X) = \overline{H^{1,0}(X)}$.

Then we have an isomorphism of real vector spaces

$$\psi : H^1(X,\mathbb{R}) \to H^{0,1}(X)$$

obtained by the composition of the inclusion $H^1(X,\mathbb{R}) \subset H^1(X,\mathbb{C})$ and the projection $H^1(X,\mathbb{C}) \to H^{0,1}(X)$ given by the Hodge decomposition of $H^1(X,\mathbb{C})$.

It follows that the lattice $H^1(X,\mathbb{Z}) \subset H^1(X,\mathbb{R})$ projects onto a lattice in the complex vector space $H^{0,1}(X)$. Thus identifying this last lattice with $H^1(X,\mathbb{Z})$ via the above isomorphism we have a complex torus

$$T = \frac{H^{0,1}(X)}{H^1(X,\mathbb{Z})}$$

associated to the Hodge structure on $H^1(X)$.

Now we prove that the complex torus T coincides with the group of isomorphism classes of holomorphic line bundles over X with Chern class 0.

For this recall that we have the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0,$$

then we have an associated long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathbb{Z}) \arrow[r, "\psi_0"] & H^0(X,\mathcal{O}_X) \arrow[r, "\varphi_0"] & H^0(X,\mathcal{O}_X^*) \arrow[dll, "c_0"] \\
& H^1(X,\mathbb{Z}) \arrow[r, "\psi_1"] & H^1(X,\mathcal{O}_X) \arrow[r, "\varphi_1"] & H^1(X,\mathcal{O}_X^*) \arrow[dll, "c_1"] \\
& H^2(X,\mathbb{Z}) \arrow[r] & H^2(X,\mathcal{O}_X) \arrow[r] & H^2(X,\mathcal{O}_X^*) \arrow[dll, dashed] \\
& H^n(X,\mathbb{Z}) \arrow[r] & H^n(X,\mathcal{O}_X) \arrow[r] & H^n(X,\mathcal{O}_X^*)
\end{tikzcd}$$

Recall also that the group $H^1(X,\mathcal{O}_X^*)$ can be identified with the Picard group of isomorphism classes of holomorphic line bundles L over X (see Theorem 4.49 in [29]).

Definition 2.1.15 (First Chern class homomorphism)**.** The connecting homomorphism $c_1 : H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z})$ is called the *first Chern class homomorphism* and the class $c_1(L)$ is called the *first Chern class of* L, see [29, page 162].

Definition 2.1.16 (The group $\mathrm{Pic}^0(X)$)**.** Set

$$\mathrm{Pic}^0(X) = \mathsf{Ker}(c_1 : H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})),$$

for the kernel of the first Chern class homomorphism c_1. This is the group of the isomorphism classes of holomorphic line bundles over X of first Chern class zero.

Proposition 2.1.17. *$T = \mathrm{Pic}^0(X)$.*

Proof. From the exactness of the above long sequence we have

$$\mathrm{Pic}^0(X) = \mathrm{im}(\varphi_1).$$

By the fundamental theorem of homomorphisms we have $\mathrm{im}(\varphi_1) \cong \frac{H^1(X,\mathcal{O}_X)}{\mathsf{Ker}(\varphi_1)}$ and by the exactness of the above long sequence again we have that $\mathsf{Ker}(\varphi_1) = \mathrm{im}(\psi_1)$, then

$$\mathrm{Pic}^0(X) = \frac{H^1(X, \mathcal{O}_X)}{\mathrm{im}(\psi_1)}.$$

By [29, §7.2.2] we have a natural isomorphism $H^{0,1}(X) \cong H^1(X, \mathcal{O}_X)$, so

$$\mathrm{Pic}^0(X) = \frac{H^{0,1}(X)}{\mathrm{im}(\psi_1)}.$$

So, in order to prove the proposition it is enough to prove that we can identify $\mathrm{im}(\psi_1) = \psi_1(H^1(X, \mathbb{Z}))$ with $H^1(X, \mathbb{Z})$ itself, that is, we must prove that ψ_1 is injective so $H^1(X, \mathbb{Z})$ is really isomorphic to $\psi_1(H^1(X, \mathbb{Z}))$. Indeed, by the exactness of the above long sequence we have $\mathsf{Ker}(\psi_1) = \mathrm{im}(c_0)$, so in order to prove that ψ_1 is injective we must prove that $\mathsf{Ker}(\psi_1) = \mathrm{im}(c_0) = 0$, i.e., that c_0 is the zero map or in other words that $\mathsf{Ker}(c_0) = H^0(X, \mathcal{O}_X^*)$. But since $\mathsf{Ker}(c_0) = \mathrm{im}(\varphi_0)$, by the exactness of the above long sequence, it is enough to prove that $\mathrm{im}(\varphi_0) = H^0(X, \mathcal{O}_X^*)$, i.e., that φ_0 is surjective. Assuming that X is projective we have $H^0(X, \mathcal{O}_X) = \mathbb{C}$ and $H^0(X, \mathcal{O}_X^*) = \mathbb{C}^*$ and φ_0 is the exponential map of complex numbers so it is surjective. $\square$

Remark 2.1.18. The torus T associated to the Hodge structure on $H^1(X)$ is itself a Kähler manifold, and thus admits a Hodge structure on its group $H^1(T)$, see [29, page 169].

Now let us study the relationship between the Hodge structure on $H^1(X)$ and the Hodge structure on $H^1(T)$ (see [29, §7.2.1]).

Lemma 2.1.19. *The Hodge structure on $H^1(T, \mathbb{Z})$ is dual to that of $H^1(X, \mathbb{Z})$.*

Proof. For a torus $T = \frac{V}{\Gamma}$, where V is a complex vector space, we have a natural identification

$$\Gamma = H_1(T, \mathbb{Z});$$

and

$$V^* = H^1(T, \mathbb{R}).$$

Furthermore, the holomorphic cotangent bundle of T is trivial, as its global sections are given by the complex linear forms on V, considered as holomorphic forms on V invariant under Γ. It follows that

$$H^{1,0}(T) = V^*.$$

Thus the Hodge structure on $H^1(T, \mathbb{Z})$ is dual to that of $H^1(X, \mathbb{Z})$, that is,

$$H^1(T, \mathbb{Z}) = H^1(X, \mathbb{Z})^* \text{ and } H^{1,0}(T) = H^{0,1}(X)^*.$$

□

In what follows we prove the following proposition (see [29, Propisition 7.16])

Proposition 2.1.20. *The complex torus* $T = \mathrm{Pic}^0(X)$ *of a projective smooth variety is an algebraic projective variety.*

Proof. Suppose now that X is a polarised manifold, and let L be the Lefschetz operator acting on the integral cohomology of X. Obviously, the cohomology of degree 1 is primitive, and thus the alternating intersection form

$$Q(\alpha, \beta) = \langle L^{n-1}\alpha, \beta\rangle, n = \dim(X),$$

defined on $H^1(X)$ and with integral values on $H^1(X, \mathbb{Z})$ satisfies the property that the Hermitian form $H(\alpha, \beta) = iQ(\alpha, \overline{\beta})$ is positive definite on $H^{1,0}(X)$, which is orthogonal to $H^{0,1}(X)$ for H, equivalently, this means that the form $Q \in \bigwedge^2(H^1(X, \mathbb{Z}))^*$ can be considered as an element ω of

$$\bigwedge^2(H^1(T, \mathbb{Z})^*) = \bigwedge^2(H^1(T, \mathbb{Z})) = H^2(T, \mathbb{Z}).$$

In fact, the de Rham class of ω is simply the class of the constant 2-form Ω on T obtained by extending Q by $\mathbb{R}$-linearity. If we identify $H_1(T, \mathbb{Z})$ with $H^1(X, \mathbb{Z})$, and thus $H_1(T, \mathbb{Z}) \otimes \mathbb{R}$ with $H^1(X, \mathbb{R})$, this differential form $\Omega = Q$ on $H^1(X, \mathbb{R})$.

The properties of Q then imply that the form Ω is a Kähler form on T. As the Kähler form thus defined on T is a of integral class, T is polarized manifold and Kodaira's theorem (Theorem 2.1.13) implies that T is an algebraic projective variety, see also Corollary 2.1.14. □

Definition 2.1.21. (Abelian variety) A complex torus that is also an algebraic projective variety is called an *abelian variety.*

From the Proposition 2.1.20 we have that the complex torus $T = \mathrm{Pic}^0(X)$ of a projective smooth variety X is an abelian variety.

Definition 2.1.22. (Picard variety) The Abelian variety $T = \mathrm{Pic}^0(X)$ of a smooth projective variety X is called the *Picard variety* of X.

2.2 Morphisms of Hodge structures

Let $(V_{\mathbb{Z}}, F^p V_{\mathbb{C}})$ and $(W_{\mathbb{Z}}, F^p W_{\mathbb{C}})$ be Hodge structures of weight n and $m = n + 2r$, $r \in \mathbb{Z}$ respectively.

Definition 2.2.1 (Morphism of Hodge structures)**.** A morphism of groups $\phi : V_{\mathbb{Z}} \to W_{\mathbb{Z}}$ is a *morphism of Hodge structures (of type* (r, r)*)* if the morphism $\phi : V_{\mathbb{C}} \to W_{\mathbb{C}}$ obtained by $\mathbb{C}$-linear extension, satisfies $\phi(F^p V_{\mathbb{C}}) \subset F^{p+r} W_{\mathbb{C}}$, or equivalently,

$$\phi(V^{p,q}) \subset W^{p+r,q+r}.$$

Remark 2.2.2. A morphism of Hodge structures ϕ induces a Hodge structure on $\mathsf{Ker}(\phi)$, see [29, Lemma 7.25].

Now recall the definition of a rational Hodge substructure

Definition 2.2.3 (Rational Hodge substructure)**.** Let $W_{\mathbb{Q}}$ be a rational Hodge structure. $V_{\mathbb{Q}} \subset W_{\mathbb{Q}}$ is a *rational sub-Hodge structure* if it is a $\mathbb{Q}$-vector subspace such that $V_{\mathbb{C}}$ has a Hodge decomposition induced by that of $W_{\mathbb{C}}$ (see [29, page 176]).

We have the following property of rational Hodge substructures

Lemma 2.2.4. *Let $V_{\mathbb{Q}} \subset W_{\mathbb{Q}}$ be a rational sub-Hodge structure. Then if the Hodge structure on W is polarised, then the same holds for the Hodge structure on V, and we have a decomposition as a direct sum*

$$W_{\mathbb{Q}} = V_{\mathbb{Q}} \oplus V'_{\mathbb{Q}},$$

where $V'_{\mathbb{Q}}$ is also a sub-Hodge structure of $W_{\mathbb{Q}}$.

Proof. See [29, Lemma 7.26]. □

Two important examples of morphisms of Hodge structures are the following

Definition 2.2.5 (Pullback homomorphism)**.** Let $\phi : X \to Y$ be a continuous map between two topological spaces. The homomorphism

$$\phi^* : H^k(Y, \mathbb{Z}) \to H^k(X, \mathbb{Z}),$$

of cohomology groups is called the *pullback homomorphism* induced by ϕ.

The pullback homomorphism is induced by the natural morphism of sheaves

$$\mathbb{Z}_Y \to \phi_* \mathbb{Z}_X.$$

There are other ways to define ϕ^*, see for example [29, §7.3.2].

Proposition 2.2.6. *If $\phi : X \to Y$ is a holomorphic map between Kähler manifolds then $\phi^* : H^k(Y, \mathbb{Z}) \to H^k(X, \mathbb{Z})$ is a morphism of Hodge structures.*

Proof. See [26, Corollary 1.13.] or [29, page 177]. □

Definition 2.2.7 (Gysin homomorphism)**.** Let $\phi : X \to Y$ be a morphism between two Kähler manifolds of dimension n and $n + r$ respectively. The homomorphism in cohomology groups

$$\phi_* : H^k(X, \mathbb{Z}) \to H^{k+2r}(Y, \mathbb{Z})$$

is called the *Gysin homomorphism* induced by ϕ.

The Gysin homomorphism is defined using Poincaré duality for X and Y as the morphism

$$\phi_* : H_{2n-k}(X, \mathbb{Z}) \to H_{2n-k}(Y, \mathbb{Z})$$

on singular homology groups, and ϕ_* is defined in the singular chains. For other ways to define ϕ_*, see [29, §7.3.2].

Proposition 2.2.8. *The Gysin morphism ϕ_* is a morphism of Hodge structures of bidegree (r, r), that is, it takes classes α of type (p, q) to classes $\phi_*(\alpha)$ of type $(p+r, q+r)$.*

Proof. See [29, page 179]. □

A important property for us is the following

Proposition 2.2.9. *The Gysin homomorphism ϕ_* on cohomology groups induce a Hodge structure on its kernel.*

Proof. It follows by Remark 2.2.2 since ϕ_* is a morphism of Hodge structures by Proposition 2.2.8. □

2.3 The intermediate Jacobian

The k-th intermediate Jacobian

Let X be a compact Kähler manifold.

Recall that for every $k > 0$, the Hodge filtration on $H^{2k-1}(X, \mathbb{C})$ determines the Hodge decomposition (see Remark 2.1.6), that is,

$$H^{2k-1}(X, \mathbb{C}) = F^k H^{2k-1}(X) \oplus \overline{F^k H^{2k-1}(X)},$$

then $F^k H^{2k-1}(X) \cap H^{2k-1}(X, \mathbb{R}) = \{0\}$, and the decomposition map gives an isomorphism

$$H^{2k-1}(X, \mathbb{R}) \to \frac{H^{2k-1}(X, \mathbb{C})}{F^k H^{2k-1}(X)}.$$

In consequence, the lattice $H^{2k-1}(X, \mathbb{Z})$ in $H^{2k-1}(X, \mathbb{R})$ gives a lattice in the $\mathbb{C}$-vector space $\frac{H^{2k-1}(X,\mathbb{C})}{F^k H^{2k-1}(X)}$. Identifying this last lattice with $H^{2k-1}(X, \mathbb{Z})$ via the above isomorphism we define the k-th intermediate Jacobian of a compact Kähler manifold as follows

Definition 2.3.1 (The k-th intermediate Jacobian)**.** The *k-th intermediate Jacobian* is defined by

$$J^{2k-1}(X) = \frac{H^{2k-1}(X,\mathbb{C})}{F^kH^{2k-1}(X) \oplus H^{2k-1}(X,\mathbb{Z})}.$$

More generally, we can define a complex torus for every Hodge structure of weight $2k-1$ as follows (see [29, Remark 12.3])

Definition 2.3.2 (Complex torus for Hodge structure of weight $2k-1$)**.** Let $V_{\mathbb{Z}}$ be a Hodge structure of weight $2k-1$. The complex torus associated to it is defined by

$$J^{2k-1}(V) := \frac{V_{\mathbb{C}}}{(F^kV \oplus V_{\mathbb{Z}})}.$$

This construction is functorial, in the sense that every morphism of Hodge structures $(V_{\mathbb{Z}}, F^{\bullet}V) \to (W_{\mathbb{Z}}, F^{\bullet+r}W)$ of bidegree (r,r) induces a morphism of complex tori

$$J^{2k-1}(V) \to J^{2(k+r)-1}(W).$$

The Jacobian of a smooth projective variety is an Abelian variety

Recall that given a compact Kähler manifold X we define the group

$$\mathrm{Pic}^0(X) := \mathsf{Ker}(H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}))$$

(Definition 2.1.16) and using the long exact sequence associated to the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

we proved that

$$\mathrm{Pic}^0(X) = T,$$

where $T = \frac{H^{0,1}(X)}{H^1(X,\mathbb{Z})}$ is the complex torus associated to the Hodge structure on $H^1(X,\mathbb{Z})$ (see Proposition 2.1.17).

The following proposition gives an alternative definition of the complex torus $\mathrm{Pic}^0(X)$.

Proposition 2.3.3. $J^1(X) = \mathrm{Pic}^0(X)$*, where* $J^1(X)$ *is the* 1*-th intermediate Jacobian of* X*.*

Proof. By definition for the case $k = 1$ we have

$$J^1(X) = \frac{H^1(X,\mathbb{C})}{F^1H^1(X) \oplus H^1(X,\mathbb{Z})}.$$

Now recall that there is an identification $\frac{H^1(X,\mathbb{C})}{F^1H^1(X)} = H^1(X,\mathcal{O}_X)$. Then

$$J^1(X) = \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})}.$$

By [29, §7.2.2] we have a natural isomorphism $H^{0,1}(X) \cong H^1(X,\mathcal{O}_X)$, it follows that

$$J^1(X) = \frac{H^{0,1}(X)}{H^1(X,\mathbb{Z})} = T = \mathrm{Pic}^0(X),$$

where the last equality holds by Proposition 2.1.17. $\square$

Proposition 2.3.4. *The* 1*-th intermediate Jacobian* $J^1(X)$ *of a smooth projective variety* X *is an Abelian variety.*

Proof. It follows from Proposition 2.3.3 and Proposition 2.1.20. □

Remark 2.3.5. In general, the k-th intermediate Jacobian $J^{2k-1}(X)$ is a transcendental object even if X is a smooth projective variety whose nature is much more difficult to understand than of $J^1(X)$.

Cohomology Group of a Curve and its Jacobian

Recall that a complex smooth projective curve C is an example of complex compact Kähler manifold of dimension 1 (see Example 2.1.2).

In this subsection we will prove an important isomorphism between the first cohomology group of a connected complex smooth projective curve C and the first cohomology group of its Jacobian $J(C)$. More precisely,

Lemma 2.3.6. *(Fact 3) Let* C *be a connected complex smooth projective curve. Then the homomorphism*

$$w_* : H^1(J(C), \mathbb{Z}) \to H^1(C, \mathbb{Z})$$

on cohomology groups is an isomorphism.

Proof. Since in particular C is a complex compact Kähler manifold of dimension 1, by Proposition 2.3.3 we have that $J(C) = T = \mathrm{Pic}^0(C)$, where $T = \frac{H^{0,1}(C)}{H^1(C,\mathbb{Z})}$ is the complex torus associated to the Hodge structure on $H^1(C, \mathbb{Z})$, so $J(C)$ is itself a Kähler manifold and thus admits a Hodge structure on its group $H^1(J(C))$ (see Remark 2.1.18), by Lemma 2.1.19 we have

$$H^1(J(C), \mathbb{Z}) \cong H^1(C, \mathbb{Z})^*$$

and since C is connected and oriented (see [9, C.2.1]) by Poincaré duality (see [29, Theorem 5.30]) we have that $H^1(C, \mathbb{Z})^* \cong H^1(C, \mathbb{Z})$. It follows that

$$H^1(J(C), \mathbb{Z}) \cong H^1(C, \mathbb{Z}).$$

□

Remark 2.3.7. The existence of the isomorphism in Lemma 2.3.6 is also true for connected smooth projective curves C over an arbitrary algebraically closed field of characteristic zero, that is, the homomorphism

$$w_* : H^1_{et}(J(C), \mathbb{Q}_l) \to H^1_{et}(C, \mathbb{Q}_l)$$

is an isomorphism (see [1, Remark 4]).

2.4 The Abel-Jacobi map and The Albanese map

The Abel-Jacobi map

Let X be a compact Kähler manifold of dimension n.

Definition 2.4.1 (The Abel-Jacobi map)**.** Let $\mathsf{Z}^k(X)_{\hom}$ be the group of cycles of codimension k homologous to 0 (also called cohomologous to 0), let $J^{2k-1}(X)$ be the k-th intermediate Jacobian. The *Abel-Jacobi map* is a morphism

$$\Phi_X^k : \mathsf{Z}^k(X)_{\hom} \to J^{2k-1}(X),$$

defined by $Z \mapsto \Phi_X^k(Z) = \int_\gamma$, where $\gamma \subset X$ is a differentiable chain of dimension $2n-2k+1$ such that $\partial\gamma = Z$ and $\int_\gamma \in \frac{F^{n-k}H^{2n-2k+1}(X)^*}{H_{2n-2k+1}(X,\mathbb{Z})} = J^{2k-1}(X)$, see [29, §12.1.2].

The equality

$$J^{2k-1}(X) = \frac{F^{n-k}H^{2n-2k+1}(X)^*}{H_{2n-2k+1}(X,\mathbb{Z})}$$

in the above definition holds thanks to Poincaré duality ([29, §12.1.2]).

If we want to work in terms of dimension note that for $Z \in \mathsf{Z}_l(X)_{\hom}$ we have the Abel-Jacobi invariant $\Phi_X^{n-l}(Z) \in J^{2(n-l)-1}(X)$.

Lemma 2.4.2. *If $Z \in \mathsf{Z}_l(X)_{\mathsf{rat}}$, then $\Phi_X^{n-l}(Z) = 0$ in $J^{2(n-l)-1}(X)$*

Proof. See [30, Lemma 9.19]. □

Thanks to the above lemma we prove the existence of the Abel-Jacobi (class) map in the following proposition.

Proposition 2.4.3. *There exists a unique homomorphism*

$$\mathsf{CH}_l(X)_{\hom} \to J^{2(n-l)-1}(X)$$

from the group of l-cycles on X homologous to 0 modulo rational equivalence to the complex torus $J^{2(n-l)-1}(X)$.

Proof. Consider the Abel-Jacobi map

$$\Phi_X^{n-l} : \mathsf{Z}_l(X)_{\hom} = \mathsf{Z}^{n-l}(X)_{\hom} \to J^{2(n-l)-1}(X).$$

Observe that $\mathsf{Z}_l(X)_{\mathsf{rat}}$ is a normal subgroup of $\mathsf{Z}_l(X)_{\hom}$ by Proposition 1.2.18, so the natural surjective homomorphism

$$\varphi : \mathsf{Z}_l(X)_{\hom} \to \frac{\mathsf{Z}_l(X)_{\hom}}{\mathsf{Z}_l(X)_{\mathsf{rat}}}$$

is well defined.

By Lemma 2.4.2 we have

$$\mathsf{Z}_l(X)_{\mathsf{rat}} \subset \mathsf{Ker}(\Phi_X^{n-l} : \mathsf{Z}_l(X)_{\mathrm{hom}} \to J^{2(n-l)-1}(X)),$$

then by the fundamental theorem on homomorphism, there exists a unique homomorphism

$$\frac{\mathsf{Z}_l(X)_{\mathrm{hom}}}{\mathsf{Z}_l(X)_{\mathsf{rat}}} = \mathsf{CH}_l(X)_{\mathrm{hom}} \to J^{(2n-2l)-1}(X)$$

such that the following diagram commutes

$$\begin{tikzcd} \mathsf{Z}_l(X)_{\mathrm{hom}} \arrow[r, "\Phi_X^{n-l}"] \arrow[dr, "\varphi"] & J^{2(n-l)-1}(X) \\ & \mathsf{CH}_l(X)_{\mathrm{hom}} \arrow[u] \end{tikzcd}$$

□

Remark 2.4.4. When $l = 0$, by abuse of notation, the Abel-Jacobi (class) map of Proposition 2.4.3 is usually denoted by alb_X and called the Albanese map (see [30, Theorem 10.11]), but there is another map also denoted by alb_X and called the Albanese map which we will define later.

The Abel-Jacobi Map for Divisors

Now we give an useful alternative definition of the Abel-Jacobi map Φ_X^k for the case $k = 1$, that is for the case of divisors.

Let $D \in \mathsf{Z}^1(X)$ be a divisor, $[D]$ the cohomology class of D, $\mathcal{O}_X(D)$ the holomorphic line bundle corresponding to the divisor D, and α_D the isomorphism class of $\mathcal{O}_X(D)$.

By Lelong theorem ([29, Theorem 11.33]) $[D] = c_1(\mathcal{O}_X(D))$, then $D \in \mathsf{Z}^1(X)_{\mathrm{hom}}$, i.e., $[D] = 0$ if and only if $c_1(\mathcal{O}_X(D)) = 0$, i.e., $\alpha_D \in \mathrm{Pic}^0(X) = J^1(X)$ (see Proposition 2.3.3). So, α_D is a well defined element in $J^1(X)$.

We also have the following proposition

Proposition 2.4.5. $\Phi_X^1(D) = \alpha_D$.

Proof. See [29, Proposition 12.7]. □

This proposition gives us the following characterization of the Abel- Jacobi map for the case $k = 1$.

Definition 2.4.6 (Abel-Jacobi map for divisors)**.** Let $\mathsf{Z}^1(X)_{\mathrm{hom}}$ be the group of cycles of codimension 1 homologous to 0 (also called cohomologous to 0 in [29]), let $J^1(X)$ be the 1-th intermediate Jacobian. The *Abel-Jacobi map* is a morphism defined by

$$\begin{array}{rccc} \Phi_X^1 : & \mathsf{Z}^1(X)_{\mathrm{hom}} & \to & J^1(X) \\ & D & \mapsto & \Phi_X^1(D) = \alpha_D \end{array}$$

In what follows we prove that the Abel-Jacobi class map for divisors is an isomorphism.

Definition 2.4.7 (Alternative definition of rational equivalence)**.** Let D be a divisor on X. We say that $D \sim_{\mathsf{rat}} 0$ if it is the divisor of a meromorphic function on X ([29, Definition 12.9]).

Lemma 2.4.8. *$\mathcal{O}_X(D)$ is trivial if and only if $D \sim_{\mathsf{rat}} 0$*

Proof. $\mathcal{O}_X(D)$ is trivial then the meromorphic section σ_D of $\mathcal{O}_X(D)$ whose divisor is equal to D can be seen as a meromorphic function on X thanks to the trivialization, then by Definition 2.4.7 $D \sim_{\mathsf{rat}} 0$.

Reciprocally, if $D \sim_{\mathsf{rat}} 0$ then by definition D is the divisor of a meromorphic function ϕ on X, then ϕ gives a everywhere non-zero section σ_D (whose divisor is D) of the line bundle $\mathcal{O}_X(D)$, so $\mathcal{O}_X(D)$ is trivial. □

Lemma 2.4.9. *Let D be a divisor such that $D \sim_{\mathrm{hom}} 0$ on X. Then $\Phi^1_X(D) = 0$ if and only if $D \sim_{\mathsf{rat}} 0$*

Proof. By Abel's theorem ([29, Corollary 12.8]) we have that $\Phi^1_X(D) = 0$, i.e., $\alpha_D = 0$ if and only if $\mathcal{O}_X(D)$ is trivial. By Lemma 2.4.8 this last condition is equivalent to $D \sim_{\mathsf{rat}} 0$. □

Then we get the following important theorem

Theorem 2.4.10.

$$\mathsf{CH}^1(X)_{\mathrm{hom}} \xrightarrow{\sim} J^1(X).$$

Proof. From Lemma 2.4.9 we have

$$\mathsf{Ker}(\Phi^1_X : \mathsf{Z}^1(X)_{\mathrm{hom}} \to J^1(X)) = \mathsf{Z}^1(X)_{\mathsf{rat}}.$$

Since Φ^1_X is surjective (see [29, §12.2.2]), and using the first isomorphism theorem of homomorphisms we have that

$$\mathsf{CH}^1(X)_{\mathrm{hom}} = \frac{\mathsf{Z}^1(X)_{\mathrm{hom}}}{\mathsf{Z}^1(X)_{\mathsf{rat}}} \xrightarrow{\sim} J^1(X)$$

is an isomorphism. □

Remark 2.4.11. Since $J^1(X) = \mathrm{Pic}^0(X)$ we also have that

$$\mathsf{CH}^1(X)_{\mathrm{hom}} \xrightarrow{\sim} \mathrm{Pic}^0(X).$$

As a easy consequence of the above theorem we get the following corollary which is very important for us.

Lemma 2.4.12. *(Fact 1) Let C be a smooth projective complex curve, let $J = J(C)$ be the Jacobian of the curve C. Then there exists an isomorphism*

$$alb_C : \mathsf{CH}_0(C)_{\deg=0} \to J$$

between the Chow group $\mathsf{CH}_0(C)_{\deg=0}$ of 0-cycles of degree zero on C and the Abelian variety J.

Proof. Since C is in particular a compact Kähler manifold (of dimension 1) by Theorem 2.4.10 we have that the Abel-Jacobi (class) map

$$alb_C : \mathsf{CH}^1(C)_{\text{hom}} = \mathsf{CH}_0(C)_{\text{hom}} \xrightarrow{\sim} J$$

is an isomorphism (see also Remark 2.4.4), by fact 2 (see Lemma 1.2.24) $\mathsf{CH}_0(C)_{\text{hom}} = \mathsf{CH}_0(C)_{\deg=0}$, so we get that

$$alb_C : \mathsf{CH}_0(C)_{\deg=0} \xrightarrow{\sim} J$$

is an isomorphism. Finally, since C is smooth and projective $J = \mathrm{Pic}^0(C)$ is an abelian variety see Proposition 2.1.20. $\square$

Hence we can identify $\mathsf{CH}_0(C)_{\deg=0}$ with J by means of alb_C.

Remark 2.4.13. If X is a smooth projective variety of dimension 1 over an arbitrary algebraically closed field there exists a universal pair (A, φ), that is, an abelian variety $A = \mathrm{Pic}^0(X) = J(X) = \mathrm{Alb}(X)$ and a regular homomorphism $\varphi : \mathsf{CH}_0(X)_{\deg=0} = \mathsf{A}_0(X) \to A$ satisfying the universal property (see [23, Notations]). Moreover, the regular homomorphism φ is an isomorphism, see [6] and [1, Remark 2].

The Albanese variety and the Albanese map

In order to define the Albanese map we need to remember the following theorem due to Griffiths.

Theorem 2.4.14. *(Griffiths' theorem) Let X be a compact Kähler manifold, Y a connected manifold, $t_0 \in Y$ a reference point, and $Z = \sum_i n_i Z_i \in \mathsf{Z}^k(Y \times X)$ a cycle of codimension k with each Z_i smooth or flat and such that $\mathsf{pr}_1 : Z_i \to Y$ is a submersion. Then the fibres*

$$Z_t = \sum_i n_i Z_{i,t}, \text{ where } Z_{i,t} := \mathsf{pr}_1^{-1}(t) \subset X,$$

are all homologous in X, and the map

$$\begin{array}{rccc} \phi : & Y & \to & J^{2k-1}(X) \\ & t & \mapsto & \Phi_X^k(Z_t - Z_{t_0}) \end{array}$$

is holomorphic (see [29, Theorem 12.4], see also [29, remark 12.5]).

Griffiths' theorem applied to the following particular case: let X be a connected manifold, $Y = X$, $t_0 = x_0 \in X$, $Z = \Delta \in \mathsf{Z}^n(X \times X)$, where $\Delta = \{(x, y) \in X \times X : x = y\}$ is the diagonal. Give us a holomorphic map

$$\begin{array}{rccc} alb_X: & X & \to & J^{2n-1}(X) \\ & x & \mapsto & \Phi_X^{2n-1}(x - x_0). \end{array}$$

Definition 2.4.15 (The Albanese map)**.** The map alb_X is called the *Albanese map*

Remark 2.4.16. Note that alb_X is the composition

$$\begin{array}{ccccc} X & \to & \mathsf{Z}^n(X)_{\text{hom}} & \overset{\Phi_X^{2n-1}}{\to} & J^{2n-1}(X) \\ x & \mapsto & x - x_0 & \mapsto & \Phi_X^{2n-1}(x - x_0). \end{array}$$

Definition 2.4.17 (Albanese variety)**.** The complex torus $\mathrm{Alb}(X) := J^{2n-1}(X)$ is called the *Albanese variety* of X.

Example 2.4.18. If $\dim(X) = 1$, that is, if X is a curve, then $\mathrm{Alb}(X) = J(X)$.

Property: the image $alb_X(X)$ generates the torus $\mathrm{Alb}(X)$ as a group. More precisely, for sufficiently large r, the morphism

$$\begin{array}{rccc} alb_X^r: & X^r & \to & \mathrm{Alb}(X) \\ & (x_1, \ldots, x_r) & \mapsto & \sum_i alb_X(x_i) \end{array}$$

is surjective. This property implies that if X is a projective variety then $\mathrm{Alb}(X)$ is an Abelian variety ([29, Corollary 12.12]).

There is another important characterization of the Albanese morphism:

Theorem 2.4.19. *For any holomorphic map $\psi : X \to T$ from X to a complex torus T such that $\psi(x_0) = 0$, there exists a unique morphism of complex tori $f : \mathrm{Alb}(X) \to T$ such that the following diagram*

$$\begin{tikzcd} X \arrow[r, "alb_X"] \arrow[dr, "\psi"'] & \mathrm{Alb}(X) \arrow[d, "f"] \\ & T \end{tikzcd}$$

commutes.

Remark 2.4.20. The Abel-Jacobi (class) map $\mathsf{CH}_0(X)_{\text{hom}} \to \mathrm{Alb}(X) = J^{2n-1}(X)$ (Proposition 2.4.3) is usually also denoted by alb_X and called the Albanese map, see [[30], Theorem 10.11].

Finally we show that a correspondence between two smooth compact Kähler manifolds induce a morphism of the corresponding Albanese varieties.

Let X and Y be two smooth compact Kähler manifolds, with Y connected, and $Z = \sum_i n_i Z_i \in \mathsf{Z}^r(Y \times X)$ such that each Z_i is flat over Y. Let $y_0 \in Y$ be a reference point.

By the generalization of Griffith's theorem to the flat case (see [29, remark 12.5]) there is a holomorphic map

$$\begin{array}{rccc} \phi : & Y & \to & J^{2r-1}(X) \\ & y & \mapsto & \Phi_X^r(Z_y - Z_{y_0}), \end{array}$$

where $Z_y = \sum_i n_i Z_{i,y}$, $Z_{i,y} = \mathsf{pr}_1^{-1}(y) \subset X$.

By the Theorem 2.4.19, that is, by the universality of the Albanese morphism there is a morphism of complex tori

$$\psi : \mathrm{Alb}(Y) \to J^{2r-1}(X)$$

such that the diagram

$$\begin{tikzcd} Y \arrow[r, "alb_Y"] \arrow[dr, "\phi"'] & \mathrm{Alb}(Y) \arrow[d, "\psi"] \\ & J^{2r-1}(X) \end{tikzcd}$$

commutes, that is, such that $\phi = \psi \circ alb_Y$.

Remark 2.4.21. The morphism ψ is the morphism of complex tori $[\tilde{Z}]$ induced by the morphism of Hodge structures (Definition 2.3.2)

$$[Z] : H^{2m-1}(Y, \mathbb{Z}) \to H^{2r-1}(X, \mathbb{Z}),$$

where $m = \dim(Y)$, given by the Künneth component

$$[Z]^{1,2r-1} \in H^1(Y, \mathbb{Z}) \otimes H^{2r-1}(X, \mathbb{Z})$$

of $[Z]$ ([29, Theorem 12.17]).

Note that if $r = \dim(X)$ we get:

$$\psi : \mathrm{Alb}(Y) \to \mathrm{Alb}(X) := J^{2\dim(X)-1}(X)$$

i.e. the correspondence $Z \in \mathsf{Z}^{\dim(X)}(Y \times X)$ gives a morphism on Albanese varieties of Y and X, respectively.

Chapter 3

Lefschetz Pencils and The Monodromy Argument

In this chapter we study the notion of Lefschetz pencils of hyperplane sections on an n-dimensional smooth projective variety. We start with the definition of Lefschetz pencils of hyperplane sections on an n-dimensional smooth projective variety X and we study another characterization of it in terms of the discriminant locus also called discriminant variety, then we study the local description of the topology of a Lefschetz degeneration and we obtain Corollary 3.2.8 in terms of homology groups which is used to give a qualitative description of the vanishing cohomology in degree $n-1$, next we apply this local description to a Lefschetz pencil $(X_t)_{t\in\mathbb{P}^1}$ of hyperplane sections of X which gives a morphism $f : \tilde{X} \to \mathbb{P}^1$, where $\tilde{X}$ is the blow up of X along the base locus of the pencil and such that each fiber of f can be naturally identified with a hyperplane section X_t of X, and we prove that the vanishing cohomology $H^{n-1}(X_t, \mathbb{Z})_{\mathsf{van}}$ is generated by the classes of vanishing spheres of a Lefschetz pencil passing through X_t, see Lemma 3.2.19.

In this chapter we also study the monodromy action on the cohomology of the fibres of a projective morphism. We begin with the definition of local systems, then we study the local monodromy for Lefschetz degenerations, which gives us the Picard-Lefschetz formula (Theorem 3.4.2), next we study the monodromy action associated to the smooth universal hypersurface where Zariski's theorem shows that whenever the discriminant locus of X is a hypersurface the monodromy action associated to the smooth universal hypersurface can be computed by restricting to a Lefschetz pencil, after that, whenever we are in the case where the discriminant locus of X is a hypersurface, we prove that all the vanishing cycles are conjugate under the monodromy action, and finally we use all these facts to prove the irreducibility of the monodromy action, i.e., Proposition 3.4.12 which plays an important role in the Hodge theoretical study of algebraic varieties over $\mathbb{C}$. This theorem says that there exists no nontrivial local subsystem of the local system with stalk $H^{n-1}(X_t, \mathbb{Q})_{\mathsf{van}}$. The main reference for this chapter is [30], see also [26].

3.1 Lefschetz pencils

Let X be a complex variety.

Definition 3.1.1 (Pencil of hypersurfaces on a variety)**.** Let $\mathscr{L}$ be a holomorphic line bundle on X and $|\mathscr{L}| := \mathbb{P}(H^0(X, \mathscr{L}))$. A *pencil of hypersurfaces on* X is a projective line $L \cong \mathbb{P}^1$ in $|\mathscr{L}|$.

Remark 3.1.2. (Another characterization of a pencil of hypersurfaces) Note that every element $t \in L$ of this pencil is a class of a nonzero and well defined up to a coefficient section σ_t of L. If $X_t \subset X$ denotes the hypersurface on X defined by the section σ_t we get a one to one correspondence between these hypersurfaces X_t and the points $t \in L$, we then write $(X_t)_{t\in L}$ for the *pencil of hypersurfaces of* X. Every σ_t is of the form $\sigma_t = \sigma_0 + t\sigma_\infty$ for $t \in \mathbb{C} \subset \mathbb{P}^1$.

Assume that $X \subset \mathbb{P}^N$ is a projective subvariety of $\mathbb{P}^N$ and $\mathscr{L} = \mathcal{O}_X(1)$. If the restriction map

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(X, \mathcal{O}_X(1))$$

is an isomorphism, that is, $|\mathscr{L}| := \mathbb{P}(H^0(X, \mathscr{L})) \cong \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))) = (\mathbb{P}^N)^*$, where $(\mathbb{P}^N)^*$ is the dual projective space parametrising the hyperplanes of $\mathbb{P}^N$, we have the following definition of a pencil.

Definition 3.1.3 (Pencil)**.** A *pencil* in $|\mathscr{L}| := \mathbb{P}(H^0(X, \mathscr{L}))$ is a line L in $(\mathbb{P}^N)^*$.

Definition 3.1.4 (Base locus of a pencil)**.** The *base locus* or *axis* of a pencil $(X_t)_{t\in L}$ is defined by

$$A = \bigcap_{t\in L} X_t \subset X.$$

Remark 3.1.5. Since $\sigma_t = \sigma_0 + t\sigma_\infty$ for $t \in \mathbb{C} \subset \mathbb{P}^1$, clearly A is defined by the equations: $\sigma_0 = \sigma_\infty = 0$. So $A = \bigcap_{t\in L} X_t = X_0 \cap X_\infty$ is a *complete intersection of codimension* 2 in X if the hypersurfaces X_0 and X_∞ have no common component.

Lefschetz pencil

Definition 3.1.6 (Lefschetz pencil)**.** A pencil $(X_t)_{t\in L}$ of hypersurfaces of X is called a *Lefschetz pencil* if it satisfies the following conditions:

1. The base locus A is smooth of codimension 2 in X. In particular, the hypersurfaces of the pencil are smooth along A.

2. Every hypersurface X_t has at most one ordinary double point as singularity.

In what follows we will give another characterization of Lefschetz pencils.

Let $X \subset \mathbb{P}^N$ be a variety contained in $\mathbb{P}^N$, and assume that X is not degenerate, i.e., the restriction map

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(X, \mathcal{O}_X(1))$$

is injective.

For every $t \in (\mathbb{P}^N)^*$ let H_t the hyperplane in $\mathbb{P}^N$ corresponding to t and consider the algebraic subset defined by

$$Z = \{(x,t) \in X \times (\mathbb{P}^N)^* : X_t := X \cap H_t \text{ is singular at } x\}.$$

It is known that Z is smooth and $\dim(Z) = N-1$, see [30, §2.1.1].

Definition 3.1.7 (The discriminant variety of X/discriminant locus of $(\mathbb{P}^N)^*$)**.** The image of Z via the second projection

$$\Delta_X = \mathsf{pr}_2(Z)$$

is called the *discriminant variety of* X or *discriminant locus of* $(\mathbb{P}^N)^*$.

By definition Δ_X is the set of singular hyperplane sections of X. It is known that $\dim(\Delta_X) \leq N-1$ and that $\dim(\Delta_X) = N-1$ if there exist hyperplane sections of X having an ordinary double point ([30, §2.1.1]).

Definition 3.1.8 (A special open subset of the discriminant locus)**.** The subset of Δ_X parametrizing hyperplanes H_t such that X_t has at most one ordinary double point as singularity is denoted by Δ_X^0.

Remark 3.1.9. If $\dim(\Delta_X) = N-1$ then $\Delta_X^0 \neq \emptyset$ and thus dense, since it is clearly a Zariski open set of Δ_X. Moreover, Δ_X^0 is smooth since pr_2 is an isomorphism over Δ_X^0 (see [30, page 45]).

Then we have the following characterization of Lefschetz pencils

Proposition 3.1.10. *Let X be a smooth subvariety of $\mathbb{P}^N$. Then a pencil of hyperplane sections $(X_t)_{t\in L}$ is a Lefschetz pencil if and only if one of the following two conditions is satisfied.*

1. $\dim(\Delta_X) = N-1$*, i.e., the discriminant variety of X is a hypersurface, and the corresponding line $L \subset (\mathbb{P}^N)^*$ to this pencil meets the discriminant hypersurface Δ_X transversely in the open dense set Δ_X^0.*

2. $\dim(\Delta_X) \leq N-2$ *and the corresponding line $L \subset (\mathbb{P}^N)^*$ to this pencil does not meet Δ_X.*

Proof. See [30, Proposition 2.9]. □

Remark 3.1.11. Note that if $p \in \mathbb{P}^N$ is such that $p \in A$ then it lies in every hyperplane of the pencil, and the hyperplanes of the pencil are exactly those containing the axis. Moreover, through any point $p \in \mathbb{P}^N$ such that $p \notin A$ there passes exactly one hyperplane in the pencil, see [21, Chapter 31].

Corollary 3.1.12. *If $X \subset \mathbb{P}^N$ is a smooth projective complex variety, then a generic pencil $(X_t)_{t \in L}$ of hyperplane sections of X is a Lefschetz pencil.*

Proof. See [30, Corollary 2.10]. □

Proposition 3.1.13. *If $X \subset \mathbb{P}^N$ is a smooth non-linear surface, then Δ_X is a hypersurface, that is,* $\dim(\Delta_X) = N - 1$.

Proof. See [28, Example 7.5]. □

3.2 Local and global Lefschetz theory

Local Lefschetz theory

In this section we study the topology of an ordinary singularity.

Definition 3.2.1 (Lefschetz degeneration map)**.** Let $B \subset \mathbb{C}^n$ be a ball of radius r centered at $0 \in \mathbb{C}^n$, f the function on B defined by $f(z) = \sum_i z_i^2$, and $B_t := f^{-1}(t)$ the fibre over t. The map f has values in the disk D of radius r^2 and is such that the central fibre B_0 has an ordinary double point at 0 as singularity, whereas the fibres B_t for t near 0 are smooth. The map $f : B \to D$ is called a *Lefschetz degeneration*.

For every point $t = |t|e^{i\theta} \in D^*$ ($D^* = D - \{0\}$) such that $|t| \leq r^2$, the fibre B_t contains the sphere S_t^{n-1} defined by

$$S_t^{n-1} = \{z = (z_1, \ldots, z_n) \in B : z_i = \sqrt{|t|}e^{i\theta/2}x_i, x_i \in \mathbb{R}, \sum_{1 \leq i \leq n} x_i^2 = 1\}$$

Definition 3.2.2 (Vanishing sphere)**.** The sphere S_t^{n-1} contained in the fiber B_t is called a *vanishing sphere of the family* $(B_t)_{t \in D}$.

The name of the sphere S_t^{n-1} is due to the fact that when t tends to 0 (i.e. to the singular point) the sphere tends to contract to a point.

Remark 3.2.3. The sphere S_t^{n-1} depends on the choice of coordinates and does not have any privileged orientation. However, its homology class $\delta \in H_{n-1}(B_t, \mathbb{Z})$, defined by the choice of an orientation, is well defined up to sign and is a generator of $H_{n-1}(B_t, \mathbb{Z})$.

Definition 3.2.4 (Vanishing cycle)**.** The homology class δ of the vanishing sphere S_t^{n-1} is called the *vanishing cycle* of the Lefschetz degeneration $f : B \to D$.

On the other hand, the set $B_{\leq|t|} = \{z \in B : |f(z)| \leq |t|\}$ contains the ball

$$B_t^n = \{z = (z_1, \ldots, z_n) \in B : z_i = \sqrt{|t|}e^{i/2\theta}x_i, x_i \in \mathbb{R}, \sum_{1\leq i\leq n} x_i^2 \leq 1\}$$

Definition 3.2.5 (Cone on the vanishing sphere)**.** The ball B_t^n contained in $B_{\leq|t|}$ is called the *cone on the vanishing sphere* S_t^{n-1}.

As an application of Morse theory we have

Proposition 3.2.6. *For D of small radius with respect to the radius of B, and $s \in D^*$, there exists a retraction by deformation $(H_t')_{t\in[0,1]}$ of $B_D := f^{-1}(D)$ onto the union $B_s \cup B_s^n$ of the fiber B_s with the ball B_s^n. Moreover, this retraction by deformation can be chosen so as to preserve S_D ($S_D \cong S_s \times D$, for some $s \in D$) and to be induced on S_D by a retraction $(R_{S,t})_{t\in[0,1]}$ as above.*

Proof. See [30, Proposition 2.14]. □

A global version of the above proposition states the following.

Let $f : X \to D$ is a proper holomorphic map from a n-dimensional complex variety X to a disk such that f is a submersion over the punctured disk D^* and that f has a nondegenerate critical point x_0 over $0 \in D$, that is, that is, let f be a Lefschetz degeneration.

Theorem 3.2.7. *Then there exists a retraction by deformation of X into the union*

$$X_t \bigcup_{S_t^{n-1}} B_t^n$$

of X_t with a n-dimensional ball B_t^n which is glued to X_t along a vanishing sphere $S_t^{n-1} \subset X_t$, where $t \in D^*$.

Proof. See [30, Theorem 2.16]. □

Corollary 3.2.8. *Let $i : X_t \hookrightarrow X_D$, $t \in D^*$, be the inclusion. Then*

$$i_* : H_k(X_t, \mathbb{Z}) \to H_k(X_D, \mathbb{Z})$$

is an isomorphism for $k < n-1$, and is surjective for $k = n-1$. Moreover, the kernel of i_* is generated by the class of "the" vanishing sphere S_t^{n-1} of X_t for $k = n-1$.

Proof. See [30, Corollary 2.17]. □

Global Lefschetz theory

Definition 3.2.9 (Fibration of topological spaces)**.** $\phi : Y \to X$ is called a *fibration of topological spaces* if locally on X there exists a trivialisation of ϕ, i.e., a homeomorphism

$$Y_U := \phi^{-1}(U) \cong Y_t \times U$$

over X, where U is an open neighborhood of $t \in X$.

Example 3.2.10. By Ehresmann's Theorem if X and Y are differentiable varieties and ϕ is submersive and proper, then ϕ is a fibration.

Let X be a compact complex variety of dimension n, and let $(X_t)_{t\in\mathbb{P}^1}$ be a Lefschetz pencil on X.

Consider the variety

$$\tilde{X} = \{(x,t) \in X \times \mathbb{P}^1 : x \in X_t\}.$$

Let

$$\tau = \mathsf{pr}_1|_{\tilde{X}} : \tilde{X} \to X,$$

where $\mathsf{pr}_1 : X \times \mathbb{P}^1 \to X$ is the first projection, and let

$$f = \mathsf{pr}_2|_{\tilde{X}} : \tilde{X} \to \mathbb{P}^1,$$

where $\mathsf{pr}_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1$ is the second projection.

- By definition of blowups it is clear that $\tilde{X} \xrightarrow{\tau} X$ can be identified with the blowup of X along the base locus A of the pencil.
- Each hypersurface X_t of the Lefschetz pencil (hence of X) can be naturally identified with the fibre $f^{-1}(t) \subset \tilde{X}$ of f.

Since $(X_t)_{t\in\mathbb{P}^1}$ is a Lefschetz pencil, the base locus A is smooth and thus $\tilde{X}$ is smooth, and since each fibre X_t of f has at most one ordinary double point as singularity we can in the neighborhood of each critical value of f, apply Theorem 3.2.7 as follows:

- Let $0_i \in \mathbb{P}^1$, with $i = 1, \dots, M$ be the critical values of f.

  For each i, let D_i be a small disk of $\mathbb{P}^1$ centered at 0_i, and let $\tilde{X}_{D_i} := f^{-1}(D_i)$. Then $\tilde{X}_{D_i} \to D_i$ satisfies the property of Theorem 3.2.7, so we have that $\tilde{X}_{D_i}$ retracts by deformation onto the union

  $$X_{t_i} \bigcup_{S^{n-1}_{t_i}} B^n_{t_i}$$

  of X_{t_i} with an n-dimensional ball $B^n_{t_i}$ glued to X_{t_i} along a vanishing sphere $S^{n-1}_{t_i} \subset X_{t_i}$, where $t_i \in D_i^*$.

- Assume that ∞ is not a critical value of f.

  Let $t \in \mathbb{C} = \mathbb{P}^1 - \infty$ be a regular value, and γ_i, $i = 1, \ldots, M$ be the paths in $\mathbb{C}$ joining t to t_i, not passing through the critical values 0_i and meeting only at the point t. Then

  - $\mathbb{C} = \mathbb{P}^1 - \infty$ admits a retraction by deformation onto
  $$\bigcup_{i=1}^{M} D_i \cup \gamma_i$$
  the union of the discs D_i with the paths γ_i.
  - Since f is a proper fibration above $\mathbb{C} \setminus \{0_1, \ldots, 0_M\}$, by Ehresmann's theorem $\tilde{X} - X_\infty$ admits a retraction by deformation onto
  $$\bigcup_{i=1}^{M} \tilde{X}_{\gamma_i} \cup \tilde{X}_{D_i},$$
  where $\tilde{X}_{\gamma_i} := f^{-1}(\gamma_i)$.
  - Finally, as f is a fibration above γ_i, each $\tilde{X}_{\gamma_i}$ admits a trivialization
  $$\tilde{X}_{\gamma_i} \cong X_{t_i} \times \gamma_i,$$
  above γ_i, and correspondingly, $\tilde{X}_{\gamma_i}$ admits a retraction by deformation onto X_{t_i}. Moreover this trivialization also gives a diffeomorphism between X_{t_i} and X_t.

Theorem 3.2.11. *(Homotopy type of $\tilde{X} - X_\infty$) The variety $\tilde{X} - X_\infty$ has the homotopy type of the union of X_t with n-dimensional balls glued to X_t along $(n-1)$-dimensional spheres.*

Proof. By above we have that $\tilde{X} - X_\infty$ admits a retraction by deformation onto
$$\bigcup_{i=1}^{M} \tilde{X}_{\gamma_i} \cup \tilde{X}_{D_i},$$
also for each i, $\tilde{X}_{\gamma_i}$ admits a trivialization onto $X_{t_i} \times \gamma_i$ and a retraction by deformation onto X_{t_i} which is diffeomorphic to X_t thanks to the trivialization, on the other hand, $\tilde{X}_{D_i}$ retracts by deformation onto the union
$$X_{t_i} \bigcup_{S^{n-1}_{t_i}} B^n_{t_i}.$$
Then $\tilde{X} - X_\infty$ admits a retraction by deformation onto
$$\bigcup_{i=1}^{M} X_{t_i} \cup (X_{t_i} \bigcup_{S^{n-1}_{t_i}} B^n_{t_i}),$$

and using the diffeomorphism of X_{t_i} and X_t, we get that $\tilde{X} - X_\infty$ admits a retraction by deformation onto

$$\bigcup_{i=1}^{M} X_t \bigcup_{(S_{t_i}^{n-1})'} (B_{t_i}^n)'$$

where $(S_{t_i}^{n-1})'$ (resp. $(B_{t_i}^n)'$) is the image in X_t of $S_{t_i}^{n-1}$ (resp. $B_{t_i}^n$) in X_{t_i} under the identifications $X_{t_i} \cong X_t$. □

Remark 3.2.12. Even though the homology class δ_i of the vanishing sphere $S_{t_i}^{n-1}$ on X_{t_i}, $t_i \in D_i^*$ is well-defined up to sign as the generator of the kernel of

$$H_{n-1}(X_{t_i}, \mathbb{Z}) \to H_*(\tilde{X}_{D_i}, \mathbb{Z}),$$

the class δ_i' of the corresponding sphere $(S_{t_i}^{n-1})'$ in $H_{n-1}(X_t, \mathbb{Z})$ depends on the choice of the path γ_i.

Corollary 3.2.13. *(Results for a smooth fiber) For $t \in \mathbb{P}^1 - \infty$ such that X_t is smooth, the inclusion*

$$i_t' : X_t \hookrightarrow \tilde{X} - X_\infty$$

induces an isomorphism

$$i_{t*}' : H_k(X_t, \mathbb{Z}) \to H_k(\tilde{X} - X_\infty, \mathbb{Z}),$$

for $k < n - 1$. For $k = n - 1$, i_{t*}' is surjective and the kernel of i_{t*}' is generated by the classes of vanishing spheres.

Proof. See [30, Corollary 2.20]. □

Remark 3.2.14. Note that given a pair (X, Y), where Y is a smooth hyperplane section of $X \subset \mathbb{P}^N$ there exists a Lefschetz pencil $(X_t)_{t \in \mathbb{P}^1}$ of hyperplane sections of X, of which Y is one member X_t ([30, §2.3.2]).

Vanishing cohomology and Primitive cohomology

Let Y be a compact Kähler variety of dimension m, $[\omega] \in H^2(Y, \mathbb{R})$ be a Kähler class. Then we have the operator

$$L = [\omega] \cup : H^k(Y, \mathbb{R}) \to H^{k+2}(Y, \mathbb{R}),$$

called the *Lefschetz operator*.

Definition 3.2.15 (Primitive cohomology)**.** The *primitive cohomology* is defined by

$$H^k(Y, \mathbb{R})_{\mathsf{prim}} := \mathsf{Ker}(L^{m+1-k} : H^k(Y, \mathbb{R}) \to H^{2(m+1)-k}(Y, \mathbb{R})).$$

Remark 3.2.16. For $k = m$ we have

$$H^m(Y,\mathbb{R})_{\mathsf{prim}} := \mathsf{Ker}(L : H^m(Y,\mathbb{R}) \to H^{m+2}(Y,\mathbb{R})).$$

From now on suppose that $Y \overset{j}{\hookrightarrow} X$ is a hyperplane section of a projective variety X of dimension n (hence $m = n-1$). Then we can take $[\omega] = c_1(\mathcal{O}_Y(1)) = h_Y$, and the equality

$$j^* \circ j_* = h_Y \cup : H^k(Y,\mathbb{Z}) \to H^{k+2}(Y,\mathbb{Z})$$

says that the corresponding Lefschetz operator of $[\omega]$ satisfies

$$L = j^* \circ j_* : H^k(Y,\mathbb{R}) \overset{j_*}{\to} H^{k+2}(X,\mathbb{R}) \overset{j^*}{\to} H^{k+2}(Y,\mathbb{R}).$$

Definition 3.2.17 (Vanishing cohomology)**.** For every coefficient ring R, the *vanishing cohomology* is defined by

$$H^k(Y,R)_{\mathsf{van}} = \mathsf{Ker}(j_* : H^k(Y,R) \to H^{k+2}(X,R)).$$

As a consequence of the Lefschetz theorem, the hard Lefschetz theorem and properties of the Lefschetz operator L we have

Corollary 3.2.18. *The vanishing cohomology*

$$H^k(Y,\mathbb{Q})_{\mathsf{van}} = 0 \textit{ for } k \neq \dim(Y).$$

Furthermore,

$$H^k(Y,\mathbb{Q})_{\mathsf{van}} \subset H^k(Y,\mathbb{Q})_{\mathsf{prim}}, \textit{ for } k = \dim(Y).$$

Proof. See [30, Corollary 2.25]. □

For the case $k = n-1 = \dim(Y)$, the following is an important property of the vanishing cohomology

Lemma 3.2.19. *The vanishing cohomology $H^{n-1}(Y,\mathbb{Z})_{\mathsf{van}}$ is generated by the classes of vanishing spheres of a Lefschetz pencil passing through Y.*

Proof. See [30, Lemma 2.26]. □

Lemma 3.2.20. *The vanishing cohomology $H^{n-1}(Y,\mathbb{Z})_{\mathsf{van}}$ is a Hodge substructure.*

Proof. By Proposition 2.2.8 j_* is a morphism of Hodge structures and by Remark 2.2.2 $\ker(j_*) = H^{n-1}(Y,\mathbb{Z})_{\mathsf{van}}$ has the structure of Hodge structure. □

The following proposition give us a comparison of the primitive cohomology with the vanishing cohomology

Proposition 3.2.21. *1. There is a decomposition as an orthogonal direct sum (relative to the intersection form in $H^{n-1}(Y, \mathbb{Q})$)*

$$H^{n-1}(Y, \mathbb{Q}) = H^{n-1}(Y, \mathbb{Q})_{\mathsf{van}} \oplus j^* H^{n-1}(X, \mathbb{Q}).$$

2. Similarly, there is a decomposition as an orthogonal direct sum

$$H^{n-1}(Y, \mathbb{Q})_{\mathsf{prim}} = H^{n-1}(Y, \mathbb{Q})_{\mathsf{van}} \oplus j^* H^{n-1}(X, \mathbb{Q})_{\mathsf{prim}}.$$

Proof. See [30, Lemma 2.27]. □

3.3 Monodromy of Lefschetz pencils

Local systems on topological spaces

Let X be a topological space and let A be some commutative ring with a unit.

Definition 3.3.1 (Local system of A-modules on X)**.** A *local system of A-modules on X* consists of the following data: for each $t \in X$ an A-module G_t and for any two points $t, t' \in X$ a collection of isomorphisms $\rho([\gamma]) : G_t \xrightarrow{\sim} G_{t'}$, one for each homotopy class $[\gamma]$ of paths from t to t'. A local system with fibres G_t is usually denoted by $\mathcal{G}$.

Remark 3.3.2. In the above definition, furthermore one requires that this assignment is functorial in the sense that $\rho([e_t]) = id_{G_t}$, for the class of the constant path e_t at t and that $\rho([\gamma * \gamma']) = \rho([\gamma]) \circ \rho([\gamma'])$ for two classes of composable paths. Here $*$ denotes the product of two composable paths, see [26, Definition B.32.].

Definition 3.3.3 (Constant local system)**.** The *constant local system with fibre G* is denoted by G_X.

Definition 3.3.4 (Monodromy representation associated to a local system)**.** Let (X, t) be a pointed path connected (i.e., 0-connected) topological space, the collection

$$\{\rho([\gamma]) : G_t \to G_t | \gamma \text{ a loop at } t\}$$

defines the associated *monodromy representation*

$$\begin{array}{rccc} \rho : & \pi_1(X, t) & \to & \mathrm{GL}(G_t) \\ & [\gamma] & \mapsto & \rho([\gamma]) \end{array}$$

Definition 3.3.5 (Locally constant sheaf)**.** A *locally constant sheaf $\mathcal{F}$ on X* is a sheaf with the property that for some open cover $\{U_i\}_{i \in U}$ of X, the restrictions

$$\rho_{U_i, t} : \mathcal{F}(U_i) \to \mathcal{F}_t, t \in U_i$$

are isomorphisms.

Remark 3.3.6. (In a locally simply connected space local systems are locally constant local systems) Let $\mathcal{G}$ a local system on the topological space X. If X is a locally simply connected space, i.e., locally 1-connected, then it admits a covering $\{U_i\}_{i\in I}$ by simply connected open subsets therefore for any two points $t, t' \in U_i$ there is a unique homotopy class $[\gamma]$ of paths from t to t' inside U_i, so there is a unique isomorphism

$$f_{t,t'} : G_t \overset{\sim}{\to} G_{t'}$$

defined by any path connecting t and t' in U_i. This gives a canonical trivialization of the local system $\mathcal{G}$ above U_i, say

$$\phi_i : \mathcal{G}|U_i \overset{\sim}{\to} G_{U_i}.$$

Let X be a path connected and locally simply connected space. Then we have the following property

Lemma 3.3.7. *Let X be a path connected and locally simply connected space. Then there is a one to one correspondence between locally constant sheaves of A-modules and local systems of A-modules on X.*

Proof. See [26, Lemma B.34.]. By Remark 3.3.6 note that the one to one correspondence is actually with locally constant local systems of A-modules on X. □

In a locally connected topological space we have the following alternative definition of a local systems, see [30, §3.1.1].

Definition 3.3.8 (Local system of stalk G)**.** Let X be a locally connected topological space, and let G be an abelian group. A *local system of stalk* G is a sheaf which is locally isomorphic to the constant sheaf of stalk G.

For another definition of constant sheaf see [29, Example 4.5].

Definition 3.3.9 (Local system of A-modules of stalk G)**.** Let G be an A-module. A *local system of A-modules of stalk* G is a sheaf of A-modules which is locally isomorphic, as a sheaf of A-modules, to the constant sheaf of stalk G.

The following proposition gives the nature of a local system on a connected, locally arcwise connected and simply connected topological space.

Proposition 3.3.10. *If X is connected, locally arcwise connected and simply connected, then every local system $\mathcal{G}$ of stalk G is trivial on X i.e. isomorphic to the constant sheaf G.*

Proof. See [30, Proposition 3.9]. □

The following corollary gives a relation of local systems and representations.

Corollary 3.3.11. *If X is arcwise connected and locally simply connected and $x \in X$, we have a natural bijection between the set of isomorphism classes of local systems of stalk G, and the set of representations*

$$\pi_1(X, x) \to \mathsf{Aut}(G),$$

modulo the action of $\mathsf{Aut}(G)$ *by conjugation.*

Proof. See [30, Proposition 3.0]. □

Definition 3.3.12 (Monodromy representation)**.** The representation (of $\pi_1(X, x)$)

$$\rho : \pi_1(X, x) \to \mathsf{Aut}(\mathcal{G}_x) = \mathsf{Aut}(G)$$

corresponding to a local system is called the *monodromy representation*.

Local systems associated to a fibration

Let $\phi : Y \to X$ be a fibration of topological spaces (see Definition 3.2.9) and assume that X is locally contractible. Then for sufficiently small U, the open sets $Y_U = \phi^{-1}(U)$ have the same homotopy type as the fibre $Y_u = \phi^{-1}(u)$ with $u \in U$. Therefore, using the invariance under homotopy of U, one deduces that for every ring of coefficients A, the sheaves $R^k\phi_* A$ are locally constant sheaves. Recall that $R^k\phi_*$ is the right k-th derived functor of the functor

$$\phi_* : \text{ Category of sheaves on } Y \to \text{ Category of sheaves on } X.$$

Proposition 3.3.13. *The monodromy representation*

$$\rho : \pi_1(X, x) \to \mathsf{Aut}(H^k(Y_x, A))$$

of $\pi_1(X, x)$ on the stalk $H^k(Y_x, A) = (R^k\phi_* A)_x$ of the local system $R^k\phi_* A$ is induced by homeomorphisms of the fibre Y_x.

Proof. See [30, Page 74]. □

Monodromy and Hodge structure

In what follows we study some restrictions that Hodge theory imposes on the monodromy representation.

Definition 3.3.14 (Projective morphism)**.** A morphism $\phi : Y \to X$ of complex varieties is called *projective* if there exists a holomorphic immersion

$$i : Y \hookrightarrow X \times \mathbb{P}^N$$

such that $\mathsf{pr}_1 \circ i = \phi$.

Let $\phi : Y \to X$ be a holomorphic, submersive and projective morphism of complex varieties. Then we have a monodromy representation

$$\rho : \pi_1(X, x) \to \mathsf{Aut}(H^k(Y_x, \mathbb{Z})),$$

as above for every k.

By Hogde theory we know that every group $H^k(Y_x, \mathbb{Z})$ is equipped with a Hodge structure, i.e., with a decomposition

$$H^k(Y_x, \mathbb{C}) = H^k(Y_x, \mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(Y_x)$$

into complex subspaces such that $H^{p,q}(Y_x) = \overline{H^{q,p}(Y_x)}$ (see Example 2.1.5).

The first restriction imposed by Hodge theory on ρ is

Proposition 3.3.15. *If X is quasi-projective, then the space*

$$H^k(Y_x, \mathbb{Z})^\rho := \{\alpha \in H^k(Y_x, \mathbb{Z}) | \rho(\gamma)(\alpha) = \alpha, \forall \gamma \in \pi_1(X, x)\}$$

of invariants under ρ is a Hodge substructure $H^k(Y_x, \mathbb{Z})$.

Proof. See [30, Proposition 3.14]. □

Another consequence of Hodge theory concerns the local monodromy called the quasi-unipotence theorem.

Theorem 3.3.16. *(Quasi-unipotence theorem) Let X be a punctured disk D^*, so that $\pi_1(X, x) = \mathbb{Z}$ and the monodromy group $\mathrm{im}(\rho) \subset \mathsf{Aut}(H^k(Y_x, \mathbb{Z}))$ is generated by a single element T. Then T is quasi-unipotent, i.e., there exists integers N and M such that*

$$(T^N - 1)^M = 0.$$

In fact, we can even take $M \leq k + 1$.

Proof. See [30, Proposition 3.15]. □

3.4 The Picard-Lefschetz formula and Zariski's theorem

The Picard-Lefschetz formula for a Lefschetz degeneration

In a wider context we have the following definition of a Lefschetz degeneration (Definition 3.2.1).

Definition 3.4.1 (Lefschetz degeneration)**.** Let X be a smooth n-dimensional complex variety. The map $f : X \to D$ is called a *Lefschetz degeneration* if f is proper with non-zero differential over the punctured disc D^*, and such that the fibre X_0 has an ordinary double point as its unique singularity.

Let X be a smooth n-dimensional complex variety and $f : X \to D$ a Lefschetz degeneration. Let $t \in D^*$. Since in this case $\pi_1(D^*, t) = \mathbb{Z}$, the monodromy representation

$$\rho : \pi_1(D^*, t) \to \mathsf{Aut}(H^{n-1}(X_t, \mathbb{Z}))$$

on the cohomology of the fibre $H^{n-1}(X_t, \mathbb{Z})$ is determined by $T \in \mathsf{Aut}(H^{n-1}(X_t, \mathbb{Z}))$, where T denotes the image via ρ of the generator of $\pi_1(D^*, t)$.

Let $\delta \in H^{n-1}(X_t, \mathbb{Z})$ be the cohomology class of the sphere $S_t^{n-1} \subset X_t$ defined by an orientation, and recall that δ is a generator of

$$\mathsf{Ker}(H^{n-1}(X_t, \mathbb{Z}) \cong H_{n-1}(X_t, \mathbb{Z}) \to H_{n-1}(X, \mathbb{Z})),$$

(see Corollary 3.2.8).

Recall also that the fiber X_t is a real oriented $(2n-2)$-dimensional variety, so we have the intersection form $\langle , \rangle$ on $H^{n-1}(X_t, \mathbb{Z})$.

Then we have the following important theorem

Theorem 3.4.2. *(Picard-Lefschetz Theorem) For every $\alpha \in H^{n-1}(X_t, \mathbb{Z})$ we have*

$$T(\alpha) = \alpha + \epsilon_n \langle \alpha, \delta \rangle \delta,$$

where $\epsilon_n = \pm 1$ according to the value of n.

Proof. See [30, Theorem 3.16]. □

Monodromy action associated to the smooth Universal hypersurface and Zariski's Theorem

Definition 3.4.3 (Family of projective varieties $\mathbb{P}^N$)**.** Let T be a variety. A *family of projective varieties in the projective space $\mathbb{P}^N$ with base T* is a closed subvariety $\mathscr{X}$ of the product $\mathbb{P}^N \times T$. The fibers $X_t = p_2^{-1}(t)$ over points $t \in T$ are called the *members* or *elements* of the family; the variety $\mathscr{X}$ is called the *total space* and the family is said to be *parametrized* by T.

Example 3.4.4 (The universal family). For any closed point $t \in \mathbb{P}^{N*}$ let H_t be the corresponding hyperplane in $\mathbb{P}^N$. The subset of $\mathbb{P}^N \times \mathbb{P}^{N*}$ defined by

$$\mathscr{H} = \{(x, t) \in \mathbb{P}^N \times \mathbb{P}^{N*} : x \in H_t\}$$

is a subvariety of $\mathbb{P}^N \times \mathbb{P}^{N*}$. Since the fibers over $\mathbb{P}^{N*}$, via $p_2 : \mathscr{H} \to \mathbb{P}^{N*}$, are all hyperplanes in $\mathbb{P}^N$ we think of $\mathscr{H}$ as the family of hyperplanes in $\mathbb{P}^N$ parametrized by $\mathbb{P}^{N*}$. Needless to say that the situation is symmetric, so we may also view $\mathscr{H}$, via $p_1 : \mathscr{H} \to \mathbb{P}^N$, as the family of all hyperplanes in $\mathbb{P}^{N*}$ parameterized by $\mathbb{P}^N$. $\mathscr{H}$ is called *the universal family* (see [13, Lecture 4]).

Remark 3.4.5. The adjective *universal* of $\mathscr{H}$ is due to the fact that if $\mathscr{X}_T \subset \mathbb{P}^N \times T$ is any flat family of hyperplanes (parametrized by T) then there is a unique regular map $T \to \mathbb{P}^{N*}$ such that $\mathscr{X}_T$ is the fiber product $T \times_{\mathbb{P}^{N*}} \mathscr{H}$, that is,

$$\begin{array}{ccc} \mathscr{X}_T & \longrightarrow & \mathscr{H} \\ \downarrow & & \downarrow \\ T & \longrightarrow & \mathbb{P}^{N*} \end{array}$$

Example 3.4.6 (Universal hyperplane section). Let $X \subset \mathbb{P}^N$ be a projective variety, $\mathscr{H}$ the universal hyperplane, and $p_1 : \mathscr{H} \to \mathbb{P}^d$ is the projection on the first factor. Set

$$\begin{aligned} \mathscr{C} &= \{(x,t) \in \mathscr{H} : x \in H_t \cap X\} \\ &= p_1^{-1}(X) \end{aligned}$$

By the second description $\mathscr{C}$ is a subvariety of $X \times \mathbb{P}^{d*}$. Let f be the composition of the closed embedding $\mathscr{C} \hookrightarrow \mathscr{H}$ and $p_2 : \mathscr{H} \to \mathbb{P}^{d*}$. Since the fibers over $\mathbb{P}^{N*}$, via f, are all hyperplane sections of X we think of $\mathscr{C}$ as the family of hyperplane sections of X parametrized by $\mathbb{P}^{N*}$. This family is called the *universal hyperplane section* of X, see [13, Lecture 4].

Let $X \subset \mathbb{P}^N$ be a smooth projective connected non-degenerate variety of dimension n. Let $\Delta_X = \mathsf{pr}_2(Z) \subset (\mathbb{P}^N)^*$ be the discriminant variety of X (see Definition 3.1.7), i.e., the set of singular hyperplane sections of X. An important property of Δ_X is that it is irreducible since it is the image in $(\mathbb{P}^N)^*$ of the smooth irreducible variety Z (see [30, §3.2.2]).

Let $U := (\mathbb{P}^N)^* \setminus \Delta_X$ be complement of Δ_X.

Definition 3.4.7 (Smooth universal hyperplane section)**.** Set

$$\mathscr{C}_U = \{(x,t) \in X \times U : x \in X_t = X \cap H_t\}$$

and $f_U : \mathscr{C}_U \to U$. Since the fibers over U are smooth hyperplane sections of X we think of $\mathscr{C}_U$ as the family of smooth hyperplane sections of X parametrized by U. This family is called the *smooth universal hyperplane section* of X. Note that by definition of U, f_U is a submersion.

Now in the following remark we recall an important fact about Lefschetz pencils of hyperplane sections of X.

Remark 3.4.8. Let $L \subset \mathbb{P}^{N*}$ be a Lefschetz pencil through $t \in U$, and recall that Δ_X^0 is the open dense subset of Δ_X parametrizing hyperplane sections X_t having exactly one ordinary double point (see also[18, §1.5]).

Case 1. If $\dim(\Delta_X) \leq N-2$, then the Lefschetz pencil L does not meet Δ_X at all (see Proposition 3.1.10). In this case $\pi_1(U,t) = 1$, so there is no monodromy action associated to the fibration f_U, see [30, §3.2.2].

Case 2. If $\dim(\Delta_X) = N-1$, then the Lefschetz pencil L meets Δ_X transversely in its smooth locus Δ_X^0 (see Proposition 3.1.10).

In the second case, Zariski's theorem shows that the monodromy representation

$$\rho : \pi_1(U,t) \to \mathsf{Aut}(H^k(X_t,\mathbb{Z}))$$

can be computed by restricting to a Lefschetz pencil.

Theorem 3.4.9. *(Zariski's theorem) Let $\mathcal{Y} \subset \mathbb{P}^r$ be a hypersurface, and let $U = \mathbb{P}^r \smallsetminus \mathcal{Y}$ be its complement. Then for $t \in U$ and for every projective line $L \subset \mathbb{P}^r$ passing through t which meets $\mathcal{Y}$ transversally in its smooth locus, the natural map*

$$\pi_1(L - L\cap\mathcal{Y},t) \to \pi_1(U,t)$$

is surjective.

Proof. See [30, Theorem 3.22]. □

Next we prove that if we are in the second case of the above remark, the vanishing cycles are conjugate under the monodromy action.

Assume that $\dim(\Delta_X) = N-1$, i.e., Δ_X is a hypersurface. Fix any $t \in U$, then we have the monodromy representation

$$\rho : \pi_1(U,t) \to \mathsf{Aut}(H^{n-1}(X_t,\mathbb{Z})),$$

associated to the fibration f_U.

Moreover, for every $y \in \Delta_X^0$, let $y' \in U$ be near y, contained in a disk D_y which meets Δ_X^0 transversally at y, and such that $D_y \smallsetminus \{y\} \subset U$. Then we have a vanishing cycle (of the Lefschetz degeneration $X_{D_y} \to D_y$ obtained by restricting f_U to $X_{D_y} = f_U^{-1}(D_y)$)

$$\delta_y \in H^{n-1}(X_{y'},\mathbb{Z}) = H_{n-1}(X_{y'},\mathbb{Z}), \text{ where } X_{y'} := f_U^{-1}(y')$$

i.e., the homology class of the sphere $S^{n-1}_{y'} \subset X_{y'}$ which is well defined up to sign as a generator of the kernel of the map

$$H_{n-1}(X_{y'},\mathbb{Z}) \to H_{n-1}(X_{D_y},\mathbb{Z}),$$

see Corollary 3.2.13.

Now choose a path γ from t to y' contained in U; then, by trivialising the fibration f_U over γ, we can construct a diffeomorphism $\psi : X_{y'} \cong X_t$, well-defined up to homotopy. Thus, we have a vanishing cycle

$$\delta_\gamma = \psi_*(\delta_y) \in H_{n-1}(X_t,\mathbb{Z}) = H^{n-1}(X_t,\mathbb{Z}),$$

where $\psi_* : H_{n-1}(X_{y'},\mathbb{Z}) \to H_{n-1}(X_t,\mathbb{Z})$.

Then thanks to the fact that Δ_X is irreducible we obtain the following result

Proposition 3.4.10. *All the vanishing cycles δ_γ (one for each $y \in \Delta_X^0$) constructed above (and defined up to sign) are conjugate (up to sign) under the monodromy action ρ.*

Proof. Clearly, by definition of the monodromy action if we change the path γ above by composing it with a loop γ' based at t, the morphism ψ_* becomes

$$\rho(\gamma') \circ \psi_*,$$

(where $\rho(\gamma') : H^{n-1}(X_t, \mathbb{Z}) \to H^{n-1}(X_t, \mathbb{Z})$ is an automorphism of $H^{n-1}(X_t, \mathbb{Z})$ and $\psi_* : H^{n-1}(X_{y'}, \mathbb{Z}) \to H^{n-1}(X_t, \mathbb{Z})$) so that

$$\delta_{\gamma' \cdot \gamma} = \rho(\gamma')(\delta_\gamma).$$

It thus suffices to check what happens when we change the point y. But as Δ_X is irreducible, its smooth locus Δ_X^0 is connected, so it is arcwise connected.

If $y_1 \in \Delta_X^0$ is another point, we can choose a path l from y to y_1 in Δ_X^0 and lift it to a path l' from y' to y_1' contained in the boundary of a tubular neighborhood of Δ_X^0 in $(\mathbb{P}^N)^\vee$.

Obviously, a trivialization of f_U over l' transports the vanishing cycle

$$\delta_y \in H_{n-1}(X_{y'}, \mathbb{Z})$$

to the vanishing cycle

$$\delta_{y_1} \in H_{n-1}(X_{y_1'}, \mathbb{Z}).$$

If γ is the path from t to y' and γ' is the path from t to y_1', then the loop

$$\gamma'' := (\gamma')^{-1} \cdot l \cdot \gamma$$

based at t satisfies

$$\rho(\gamma'')(\delta_\gamma) = \delta_{\gamma'}.$$

□

Corollary 3.4.11. *Let $(X_t)_{t \in \mathbb{P}^1}$ be a Lefschetz pencil of hyperplane sections of X, 0_i, $i = 1, \ldots, M$ the critical values, and $t \in \mathbb{P}^1$ a regular value. Then all the vanishing cycles $\delta_i \in H^{n-1}(X_t, \mathbb{Z})$ of the pencil are conjugate under the monodromy action of $\rho : \pi_1(\mathbb{P}^1 \smallsetminus \{0_1, \ldots, 0_M\}, t) \to \mathsf{Aut}(H^{n-1}(X_t, \mathbb{Z}))$.*

Proof. It follows from Theorem 3.4.9 and the proposition above. □

Next we prove that the vanishing cohomology of a smooth hyperplane section is stable under the monodromy action associated to f_U.

Proposition 3.4.12. *Let X_t be a smooth hyperplane section of X. Then the monodromy action*

$$\rho : \pi_1(U,t) \to \mathsf{Aut}(H^{n-1}(X_t,\mathbb{Q})),$$

associated to the fibration f_U, leaves $H^{n-1}(X_t,\mathbb{Q})_{\mathsf{van}}$ stable.

Proof. We have an inclusion

$$\mathcal{X}_U \to U \times X$$

of fibrations $f_U : \mathcal{X}_U \to U$ and $\mathsf{pr}_1 : U \times X \to U$ over U which gives a morphism of local systems

$$J_* : R^{n-1}\phi_*\mathbb{Q} \to R^{n+1}\mathsf{pr}_{1*}\mathbb{Q}$$

whose value on the stalk at the point t is the map

$$j_* : H^{n-1}(X_t,\mathbb{Q}) \to H^{n+1}(X,\mathbb{Q}).$$

Thus, we have a local subsystem $\mathsf{Ker}J_*$ whose stalk at the point t is $\mathsf{Ker}(j_*)$.

The monodromy ρ preserves the stalks of this local sub-system, i.e., it leaves $\mathsf{Ker}(j_*)$ stable. □

Definition 3.4.13 (Irreducible action)**.** The action of a group G on a vector space E is said to be *irreducible* if every vector subspace $F \subset E$ stable under G is equal to $\{0\}$ or E.

Next we prove that there is no non-trivial local subsystem of the local system with stalk $H^{n-1}(X_t,\mathbb{Q})_{van}$.

Theorem 3.4.14. *Let $(X_t)_{t\in\mathbb{P}^1}$ be a Lefschetz pencil of hyperplane sections of X, 0_i, $i = 1,\ldots,M$ the critical values, and $t \in \mathbb{P}^1$ a regular value. Then the monodromy action*

$$\rho : \pi_1(U,t) \to \mathsf{Aut}(H^{n-1}(X_t,\mathbb{Q})_{van})$$

is irreducible.

Proof. Let $V = \mathbb{P}^1 - \{0_1,\ldots,0_M\}$. By Zariski's Theorem (Theorem 3.4.9) it suffices to prove the irreducibility of the monodromy action

$$\rho_V : \pi_1(V,t) \to \mathsf{Aut}(H^{n-1}(X_t,\mathbb{Q})_{\mathsf{van}}),$$

obtained by restringing us to the Lefschetz pencil $f_{\mathbb{P}^1} : \mathscr{C}_{\mathbb{P}^1} \to \mathbb{P}^1$ passing through t of the hypothesis.

For each 0_i with $i = 1,\ldots,M$, consider the small disk $D_i \subset \mathbb{P}^1$ centered at 0_i, $t_i \in D_i^*$ and γ_i the path joining t to t_i. Let $\delta_i' \in H_{n-1}(X_{t_i},\mathbb{Q}) = H^{n-1}(X_{t_i},\mathbb{Q})$ be the vanishing cycle (the homology class of the vanishing sphere $S_{t_i}^{n-1} \subset X_{t_i}$) which is well defined up to sign as a generator of $\mathsf{Ker}(H^{n-1}(X_{t_i},\mathbb{Q}) \to H^{n-1}(X_{\Delta_i},\mathbb{Q}))$ and recall that by trivialising the fibration $f_{\mathbb{P}^1}$ over γ_i we can construct a diffeomorphism $X_{t_i} \cong X_t$, so we have a vanishing cycle $\delta_i \in H^{n-1}(X_t, \mathbb{Q})$ which is image of δ_i' via the diffeomorphism. By Lemma 3.2.19 the vanishing cohomology $H^{n-1}(X_t, \mathbb{Q})_{\mathsf{van}}$ is generated by these vanishing cycles δ_i, $i = 1, \ldots, M$, of the Lefschetz pencil.

On the other hand, let $\tilde{\gamma}_i$ be the loop in V based at t such that $\tilde{\gamma}_i$ is equal to γ_i until t_i winds around the disk D_i once in the positive direction, and then returns to t via γ_i^{-1}. Recall that these loops $\tilde{\gamma}_i, i = 1, \ldots, M$ generate $\pi_1(V, t)$ and note that the image of the loops $\tilde{\gamma}_i$ via ρ_V are elements in $\mathsf{Aut}(H^{n-1}(X_t, \mathbb{Q})_{\mathsf{van}})$, that is, $\rho_V(\tilde{\gamma}_i) : H^{n-1}(X_t, \mathbb{Q})_{\mathsf{van}} \to H^{n-1}(X_t, \mathbb{Q})_{\mathsf{van}}$ are automorphisms.

Let $F \subset H^{n-1}(X_t, \mathbb{Q})_{\mathsf{van}}$ be a nontrivial vector subspace which is stable under the monodromy action ρ_V. We must prove that $F = H^{n-1}(X_t, \mathbb{Q})_{\mathsf{van}}$ or $F = \{0\}$.

Let $0 \neq \alpha \in F$. Since by Proposition 3.2.21, $\langle , \rangle$ is non-degenerate on $H^{n-1}(X_t, \mathbb{Q})_{\mathsf{van}}$ there exists $i \in \{1, \ldots, M\}$ such that

$$\langle \alpha, \delta_i \rangle \neq 0.$$

By the Picard-Lefschetz formula (Theorem 3.4.2) for $\alpha \in F \subset H^{n-1}(X_t, \mathbb{Q})_{\mathsf{van}}$ one has

$$\rho_V(\tilde{\gamma}_i)(\alpha) = \alpha \pm \langle \alpha, \delta_i \rangle \delta_i,$$

or equivalently,

$$\rho_V(\tilde{\gamma}_i)(\alpha) - \alpha = \pm \langle \alpha, \delta_i \rangle \delta_i.$$

Since, by assumption, F is a vector subspace of $H^{n-1}(X_t, \mathbb{Q})_{\mathsf{van}}$ which is stable under the monodromy action ρ_V (so $\rho_V(\tilde{\gamma}_i)(\alpha) \in F$) we have

$$\rho_V(\tilde{\gamma}_i)(\alpha) - \alpha = \pm \langle \alpha, \delta_i \rangle \delta_i \in F.$$

Then

$$\delta_i \in F.$$

But, Corollary 3.4.11 shows that all the vanishing cycles are conjugate under the monodromy action, so F, which is stable under the monodromy action, must contain all the vanishing cycles. Thus

$$F = H^{n-1}(X_t, \mathbb{Q})_{\mathsf{van}}.$$

□

Corollary 3.4.15. *Let $f_U : \mathcal{X}_U \to U$ be the universal smooth hypersurface. Then there exists no local subsystem of $R^{n-1}\phi_* \mathbb{Q}_{van}$ which is non-trivial, where $n = \dim(X)$.*

Proof. See [30, Corollary 3.28]. □

Chapter 4

The Gysin Kernel

In this chapter we present and prove the main result of the thesis called the *theorem on the Gysin kernel* (Theorem 4.1.1). More precisely, let S be a connected smooth projective surface over $\mathbb{C}$. Let Σ be the complete linear system of a very ample divisor D on S of dimension say d. Let $\mathbb{P}^{d*}$ be the dual projective space of $\mathbb{P}^d$ parametrizing hyperplanes in $\mathbb{P}^d$ and let $\overline{\eta}$ be the geometric generic point of $\mathbb{P}^{d*}$. Let Δ_S be the discriminant variety of S also called the discriminant locus of $\Sigma \cong \mathbb{P}^{d*}$ parametrizing singular hyperplane sections of S and $U = \Sigma \setminus \Delta_S$ its complement of smooth hyperplane sections of S. For any closed point $t \in \Sigma$, let H_t be the hyperplane in $\mathbb{P}^d$ corresponding to t, $C_t = H_t \cap S$ the corresponding hyperplane section of S, r_t the closed embedding of C_t into S, and r_{t*} the Gysin homomorphism on Chow groups from $\mathsf{CH}_0(C_t)_{\deg=0}$ to $\mathsf{CH}_0(S)_{\deg=0}$, induced by r_t, whose kernel is denoted by G_t and called the ***Gysin kernel*** associated to the hyperplane section C_t. We prove the theorem on the Gysin kernel which has the following three items. **Item a)** (see also Theorem A in [1], and [30, page 304]) states that the Gysin kernel G_t associated to a smooth hyperplane section C_t, i.e., with $t \in U$, is a countable union of translates of an abelian subvariety A_t inside $B_t \subset J_t$, where A_t is the unique irreducible component passing through zero of the irredundant decomposition of G_t and B_t is the abelian subvariety of the Jacobian J_t of the curve C_t, corresponding to the Hodge substructure on $H^1(C_t, \mathbb{Z})_{\mathsf{van}}$, the kernel of the Gysin homomorphism from $H^1(C_t, \mathbb{Z})$ to $H^3(S, \mathbb{Z})$ induced by r_t. **Item b)** (see also Theorem B in [1]) states the existence of a c-open subset U_0 in U such that A_t with $t \in U_0$ has two possibilities and also the same behaviour. More precisely, there exits a c-open subset U_0 in U such that either $A_{\overline{\eta}} = 0$, in which case $A_t = 0$ for all $t \in U_0$, or $A_{\overline{\eta}} = B_{\overline{\eta}}$, in which case $A_t = B_t$ for all $t \in U_0$. **Item c)** states that if we are in the case where the discriminant locus Δ_S is a hypersurface then for every t in U, A_t has only two possibilities but not necessarily the same behaviour. More precisely, for every $t \in U$ we have that $A_t = 0$ or $A_t = B_t$.

4.1 A Theorem on the Gysin kernel

Let S be a connected smooth projective surface over $\mathbb{C}$, D a very ample divisor on S (see Definition 1.4.21) and $\mathcal{O}_S(D)$ its corresponding very ample invertible sheaf on S. Let $\Sigma = |D| = |\mathcal{O}_S(D)|$ be the complete linear system of D on S (see Definition 1.4.24), $d = \dim(\Sigma)$, and

$$\phi_\Sigma : S \hookrightarrow \mathbb{P}^d$$

the closed embedding of S on $\mathbb{P}^d$, induced by Σ. Note in particular that since D is very ample, S is not degenerate, i.e., it is not contained in any hyperplane in $\mathbb{P}^d$ (see 1.4.31).

Let $\mathbb{P}^{d*}$ be the dual projective space of $\mathbb{P}^d$ parametrizing hyperplanes in $\mathbb{P}^d$ and let $\overline{\eta}$ be its geometric generic point. Recall that by definition the linear system Σ can be identified with $\mathbb{P}^{d*}$.

For any closed point $t \in \Sigma = \mathbb{P}^{d*}$, let H_t be the hyperplane in $\mathbb{P}^d$ corresponding to t, $C_t = H_t \cap S$ the corresponding hyperplane section of S, and

$$r_t : C_t \hookrightarrow S$$

the closed embedding of the curve C_t into S.

Let

$$\Delta_S = \{t \in \Sigma : C_t \text{ is singular}\}.$$

the subset in Σ parametrizing singular hyperplane sections of S and called the discriminant variety of S also called the discriminant locus of Σ (see Definition 3.1.7).

Let

$$U = \Sigma \setminus \Delta_S = \{t \in \Sigma : C_t \text{ is smooth}\},$$

be the complement of Δ_S parametrizing smooth hyperplane sections of S.

For any closed point $t \in U$, let

$$r_{t*} : H^1(C_t, \mathbb{Z}) \to H^3(S, \mathbb{Z})$$

be the Gysin homomorphism on cohomology groups induced by r_t (see Definition 2.2.7). Recall that $H^1(C_t, \mathbb{Z})$ and $H^3(S, \mathbb{Z})$ carries a weight 1 and 3 Hodge structure respectively (see Example 2.1.5) and that r_{t*} is a morphism of Hodge structures of bidegree $(1, 1)$ (see Proposition 2.2.8). Let $H^1(C_t, \mathbb{Z})_{\mathsf{van}}$ be the kernel of the above Gysin homomorphism r_{t*}, it is called the *vanishing cohomology* (see Definition 3.2.17) and it carries Hodge structure induced by the morphism of Hodge structures r_{t*} (see Lemma 3.2.20).

For any closed point $t \in U$, let $J_t = J(C_t)$ be the Jacobian of the curve C_t. Recall that J_t is the complex torus associated to the Hodge structure of weight 1 on $H^1(C_t, \mathbb{Z})$ (see Proposition 2.3.3) and since C_t is smooth and projective it is an Abelian variety (see Proposition 2.3.4). Let B_t be the abelian subvariety of the abelian variety J_t associated or corresponding to the Hodge substructure on $H^1(C_t, \mathbb{Z})_{\mathsf{van}}$.

For any closed point $t \in U$, let $\mathsf{CH}_0(C_t)_{\deg=0}$ be the Chow group of 0-cycles of degree zero on C_t (see Definition 1.3.6).

For any closed point $t \in U$, let

$$r_{t*} : \mathsf{CH}_0(C_t)_{\deg=0} \to \mathsf{CH}_0(S)_{\deg=0}$$

be the Gysin homomorphism on the Chow groups of 0-cycles of degree zero of C_t and S respectively, induced by r_t (see Definition 1.3.2), whose kernel

$$G_t = \mathsf{Ker}(r_{t*})$$

will be called the *Gysin kernel* associated to the hyperplane section C_t.

Theorem 4.1.1 (A theorem on the Gysin kernel)**.** *Let S, Δ_S, U, $\overline{\eta}$, G_t, B_t and J_t be as above. Then*

a) For every $t \in U$ there is an abelian subvariety A_t of $B_t \subset J_t$ such that

$$G_t = \bigcup_{countable} translates\ of\ A_t$$

b) For very general $t \in U$ either, $A_t = 0$ or $A_t = B_t$.

For the above we mean:

There exits a c-open subset $U_0 \subset U$ such that either $A_{\overline{\eta}} = 0$, in which case $A_t = 0$ for all $t \in U_0$, or $A_{\overline{\eta}} = B_{\overline{\eta}}$, in which case $A_t = B_t$ for all $t \in U_0$.

c) Assume that Δ_S is an hypersurface. Then for every $t \in U$, $A_t = 0$ or $A_t = B_t$.

Remark 4.1.2. In item b) note that if $A_t = 0$ then it follows immediately by item a) that G_t is countable and if $A_t = B_t$ and then it follows immediately by item a) that G_t is a countable union of translates of B_t. The same thing is true for item c).

Remark 4.1.3. More precisely, see Definition 1.3.12, the phrase "*For very general $t \in U$ either, $A_t = 0$ or $A_t = B_t$*" in item b) of theorem Theorem 4.1.1 can be stated as "*Either, for very general $t \in U$, $A_t = 0$ or , for very general $t \in U$, $A_t = B_t$*".

Proof of item a) of Theorem 4.1.1. Recall that $U = \Sigma \setminus \Delta_S$ is the open subset of $\Sigma = \mathbb{P}^{d*}$ parametrizing smooth hyperplane sections of S.

Let $t \in U = \Sigma \setminus \Delta_S$ be any (closed) point in U, then the corresponding curve C_t is a smooth (hence connected, see [11]) curve in S and

$$r_t : C_t \hookrightarrow S$$

is the closed embedding of the smooth connected curve C_t into S.

For each natural number d, let $\mathsf{Sym}^d(C_t)$ be the d-th symmetric product of the curve C_t, $\mathsf{Sym}^d(S)$ the d-th symmetric product of the surface S (see Definition 1.3.13),

$$\mathsf{Sym}^d(r_t) : \mathsf{Sym}^d(C_t) \to \mathsf{Sym}^d(S)$$

the morphism from the d-th symmetric product of the curve C_t to the d-th symmetric product of the surface S, induced by r_t, and

$$\mathsf{Sym}^{d,d}(r_t) : \mathsf{Sym}^{d,d}(C_t) = \mathsf{Sym}^d(C_t) \times \mathsf{Sym}^d(C_t) \to \mathsf{Sym}^{d,d}(S) = \mathsf{Sym}^d(S) \times \mathsf{Sym}^d(S).$$

Then we have the following commutative diagram

$$\begin{array}{ccc} \mathsf{Sym}^{d,d}(C_t) & \xrightarrow{\mathsf{Sym}^{d,d}(r_t)} & \mathsf{Sym}^{d,d}(S) \\ \Big\downarrow{\scriptstyle \theta_d^{C_t}} & & \Big\downarrow{\scriptstyle \theta_d^{S}} \\ \mathsf{CH}_0(C_t)_{\deg=0} & \xrightarrow{r_{t*}} & \mathsf{CH}_0(S)_{\deg=0} \end{array} \tag{4.1}$$

where $\theta_d^{C_t}$ and θ_d^S are the set-theoretic maps of Definition 1.3.14 (see also Remark 1.3.15), r_{t*} is the Gysin homomorphism on Chow groups of 0-cycles of degree 0 induced by r_t.

Now recall that by Lemma 2.4.12 (fact 1) and Lemma 1.2.24 (fact 2) there exists an isomorphism

$$alb_{C_t} : \mathsf{CH}_0(C_t)_{\deg=0} \to J_t =: \mathrm{Alb}(C_t),$$

where $J_t = J(C_t)$ is the Jacobian of the curve C_t and $\mathrm{Alb}(C_t)$ is the Albanese variety of C_t and alb_{C_t} is the Albanese map, then, by Definition 1.3.25, $\mathsf{CH}_0(C_t)_{\deg=0}$ is representable, or equivalently

$$\theta_d^{C_t} : \mathsf{Sym}^{d,d}(C_t) \to \mathsf{CH}_0(C_t)_{\deg=0}$$

is surjective for sufficiently large d, by Definition 1.3.24. This implies that the Gysin kernel is of the form

$$G_t = \theta_d^{C_t}[(\theta_d^S \circ \mathsf{Sym}^{d,d}(r_t))^{-1}(0)].$$

Indeed, by the commutativity of the diagram (4.1) we have

$$(r_{t*} \circ \theta_d^{C_t})^{-1}(0) = (\theta_d^S \circ \mathsf{Sym}^{d,d}(r_t))^{-1}(0),$$

then

$$\theta_d^{C_t}[(r_{t*} \circ \theta_d^{C_t})^{-1}(0)] = \theta_d^{C_t}[(\theta_d^S \circ \mathsf{Sym}^{d,d}(r_t))^{-1}(0)],$$

then by properties of the inverse of a composition and by the surjectivity of $\theta_d^{C_t}$ we get

$$r_{t*}^{-1}(0) = \theta_d^{C_t}[(\theta_d^S \circ \mathsf{Sym}^{d,d}(r_t))^{-1}(0)],$$

i.e.,

$$G_t = \theta_d^{C_t}[(\theta_d^S \circ \mathsf{Sym}^{d,d}(r_t))^{-1}(0)].$$

On the other hand, by Lemma 1.3.26 the preimage of 0

$$(\theta_d^S)^{-1}(0)$$

is a countable union of Zariski closed subsets in $\mathsf{Sym}^{d,d}(S)$. It follows that

$$(\theta_d^S \circ \mathsf{Sym}^{d,d}(r_t))^{-1}(0)$$

is the countable union of Zariski closed subsets in $\mathsf{Sym}^{d,d}(C_t)$.

Now for each d, consider the composition

$$\mathsf{Sym}^{d,d}(C_t) \overset{\theta_d^{C_t}}{\to} \mathsf{CH}_0(C_t)_{\deg=0} \overset{alb_{C_t}}{\to} J_t,$$

by Lemma 1.3.21 it follows that the set-theoretic map $\theta_d^{C_t}$ from the algebraic variety $\mathsf{Sym}^{d,d}(C_t)$ to $\mathsf{CH}_0(C_t)_{\deg=0}$ is regular and by Lemma 1.3.23 the composition $alb_{C_t} \circ \theta_d^{C_t}$ is a morphism of varieties. Since these varieties are projective, this composition is proper (so it takes closed subsets to closed subsets).

It follows that

$$alb_{C_t}(G_t) = alb_{C_t}(\theta_d^{C_t}[(\theta_d^S \circ \mathsf{Sym}^{d,d}(r_t))^{-1}(0)]) = alb_{C_t} \circ \theta_d^{C_t}[(\theta_d^S \circ \mathsf{Sym}^{d,d}(r_t))^{-1}(0)]$$

is also a countable union of Zariski closed subsets in the abelian variety J_t.

Now since $alb_{C_t}(G_t)$ is a countable union of algebraic varieties over $\mathbb{C}$ (which is uncountable), $alb_{C_t}(G_t)$ admits a unique irredundant decomposition inside the abelian variety J_t, see Lemma 1.3.9. Using the isomorphism alb_{C_t} we can identify $alb_{C_t}(G_t)$ with G_t and write

$$G_t = \bigcup_{n \in \mathbb{N}} (G_t)_n$$

for the irredundant decomposition of G_t in $J_t \simeq \mathsf{CH}_0(C_t)_{\deg=0}$.

On the other hand, note that G_t being the kernel of the Gysin homomorphism on Chow groups $r_{t*} : \mathsf{CH}_0(C_t)_{\deg=0} \to \mathsf{CH}_0(S)_{\deg=0}$ it is a subgroup in $\mathsf{CH}_0(C_t)_{\deg=0}$, hence $alb_{C_t}(G_t)$ its image in J_t via the isomorphism alb_{C_t} (which in particular is a homomorphism), is also a subgroup in J_t.

As J_t is an abelian variety and $G_t \subset J_t$ a subgroup which can be represented as a countable union of Zariski closed subsets in J_t, the irredundant decomposition of G_t contains a unique irreducible component passing through 0 which is an abelian subvariety of J_t (see Lemma 1.3.10). After renumbering of the components, we may assume that this component is $(G_t)_0$.

It is clear that for any $x \in G_t$, the set $x + (G_t)_0$ is an irreducible Zariski closed subset (just translation of a Zariski closed subset) in G_t, and that we can write

$$G_t = \bigcup_{x \in G_t} (x + (G_t)_0).$$

Ignoring each set $x + (G_t)_0$ inside $y + (G_t)_0$, for $x, y \in G_t$, we get a subset $\Xi_t \subset G_t$ such that

$$G_t = \bigcup_{x \in \Xi_t} (x + (G_t)_0)$$

which is an irredundant decomposition of G_t.

Now we claim that Ξ_t is countable. Indeed, for any $x, y \in \Xi_t$, $x+(G_t)_0$ and $y+(G_t)_0$ are irreducible and not contained one in another. Now observe that since $x + (G_t)_0$ is irreducible and G_t is a subgroup, then $x+(G_t)_0 \subset (G_t)_n$ for some n, because otherwise if $x+(G_t)_0$ is not contained in $(G_t)_n$ for every $n \in \mathbb{N}$, then $x+(G_t)_0$ would be the union of the closed subsets of the form $(x + (G_t)_0) \cap (G_t)_n$ each of which is not $x + (G_t)_0$, contradicting Lemma 1.3.7. It follows that $(G_t)_0 \subset -x+(G_t)_n$. Similarly, we can prove that $-x + (G_t)_n$ is contained in $(G_t)_l$ for some $l \in \mathbb{N}$. Then $(G_t)_0 = (G_t)_l$, that is, $(G_t)_0 \subset -x + (G_t)_n \subset (G_t)_0$, so $-x + (G_t)_n = (G_t)_0$, i.e., $x + (G_t)_0 = (G_t)_n$ for each $x \in \Xi_t$. It means that Ξ_t is countable.

Taking $A_t = (G_t)_0$, until now we have proved that there is there is an Abelian variety $A_t \subset J_t$ such that

$$G_t = \bigcup_{x \in \Xi_t} (x + A_t),$$

where $\Xi_t \subset G_t$ is a countable subset. Equivalently, we can write as follows: there is an Abelian variety $A_t \subset J_t$ such that

$$G_t = \bigcup_{\text{countable}} \text{translates of } A_t.$$

To complete the proof of item (a) we next show that A_t is contained in B_t.

Let

$$i : A_t \hookrightarrow J_t$$

be the closed embedding of A_t into J_t.

Fix an ample line bundle $\mathcal{L}_t$ on J_t, and let $\mathcal{L}'_t$ be the pullback of $\mathcal{L}_t$ to A_t under the embedding i. Then we have an homomorphism

$$\lambda_{\mathcal{L}'_t} : A_t \to A_t^\vee$$

from the abelian subvariety A_t to its dual induced by $\mathcal{L}'_t$, see [20, Chapter 8]. By Remark 8.7 in [20] we have that $\dim(A_t) = \dim(A_t^\vee)$. Then we have the Gysin homomorphism

$$(\lambda_{\mathcal{L}'_t})_* : H^1(A_t, \mathbb{Z}) \to H^1(A_t^\vee, \mathbb{Z})$$

on cohomology groups induced by $\lambda_{\mathcal{L}'_t}$ (see Definition 2.2.7).

Let

$$i^\vee : J_t^\vee \hookrightarrow A_t^\vee$$

be the homomorphism on dual abelian varieties (see [20, Chapter 9]) induced by the closed embedding

$$i : A_t \hookrightarrow J_t,$$

this induces the pullback homomorphism

$$(i^\vee)^* : H^1(A_t^\vee, \mathbb{Z}) \to H^1(J_t^\vee, \mathbb{Z})$$

on cohomology groups associated to $i^\vee$.

Let

$$\lambda_{\mathcal{L}_t} : J_t \to J_t^\vee$$

be the homomorphism from the abelian variety J_t to its dual induced by $\mathcal{L}_t$, see [[20], Chapter 8]. Then we have the pullback homomorphism

$$(\lambda_{\mathcal{L}_t})^* : H^1(J_t^\vee, \mathbb{Z}) \to H^1(J_t, \mathbb{Z})$$

on cohomology groups induced by $\lambda_{\mathcal{L}_t}$ (see Definition 2.2.5).

From the above we get an injective homomorphism on cohomology groups ζ_t via the following commutative diagram

$$\begin{array}{ccc} H^1(A_t, \mathbb{Z}) & \xrightarrow{\zeta_t} & H^1(J_t, \mathbb{Z}) \\ \Big\downarrow{\scriptstyle (\lambda_{\mathcal{L}'_t})_*} & & {\scriptstyle (\lambda_{\mathcal{L}_t})^*}\Big\uparrow \\ H^1(A_t^\vee, \mathbb{Z}) & \xrightarrow{(i^\vee)^*} & H^1(J_t^\vee, \mathbb{Z}) \end{array}$$

Let

$$w_{t*} : H^1(J_t, \mathbb{Z}) \to H^1(C_t, \mathbb{Z})$$

be the isomorphism given by Lemma 2.3.6 (fact 3).

By Proposition 14 in [1] the image of the composition

$$H^1(A_t, \mathbb{Z}) \overset{\zeta_t}{\to} H^1(J_t, \mathbb{Z}) \overset{w_{t*}}{\to} H^1(C_t, \mathbb{Z})$$

is contained in the kernel of the Gysin homomorphism on cohomology groups

$$H^1(C_t, \mathbb{Z})_{\mathsf{van}} = \mathsf{Ker}(H^1(C_t, \mathbb{Z}) \overset{r_{t*}}{\to} H^3(S, \mathbb{Z})),$$

i.e.,

$$(w_{t*} \circ \zeta_t)(H^1(A_t, \mathbb{Z})) \subset H^1(C_t, \mathbb{Z})_{\mathsf{van}}. \tag{4.2}$$

Now recall that B_t is the abelian subvariety of J_t corresponding to the Hodge substructure on $H^1(C_t, \mathbb{Z})_{\mathsf{van}}$, i.e.,

$$B_t = \frac{H^{0,1}(X) \cap H^1(C_t, \mathbb{Z})_{\mathsf{van}}}{H^1(X, \mathbb{Z}) \cap H^1(C_t, \mathbb{Z})_{\mathsf{van}}}$$

so $H^1(B_t,\mathbb{Z}) \cong (H^1(C_t,\mathbb{Z})_{\mathsf{van}})^* \cong H^1(C_t,\mathbb{Z})_{\mathsf{van}}$ (the composition of these isomorphisms is w_{t*}, see proof of Lemma 2.3.6). On the other hand, since w_{t*} is an isomorphism and ζ_t is injective we can identify $(w_{t*}\circ\zeta_t)(H^1(A_t,\mathbb{Z}))$ with $H^1(A_t,\mathbb{Z})$. Then by the inclusion (4.2), we get

$$H^1(A_t,\mathbb{Z}) \subset H^1(B_t,\mathbb{Z}),$$

so $A_t \subset B_t$. □

Remark 4.1.4. Note that the proof of item a) of Theorem 4.1.1 also holds any smooth hyperplane section C of a surface S over and uncountable algebraically closed field k of characteristic zero and in the adequate context.

4.2 On the proof of item b)

In order to prove item b) we will use the following lemmas.

Existence of a c-open subset of an integral scheme

We start this chapter with the approach to prove the existence of a c-open subset of an integral scheme such that the residue field of each point in this c-open is isomorphic to the residue field of the geometric generic point of the integral scheme, this isomorphism of fields then induce a scheme-theoretic isomorphism between each point of the c-open and the geometric generic point of the integral scheme, and given a family over this integral scheme the above scheme-theoretic isomorphism of points induces an isomorphism between the corresponding fibers of the family as schemes over $\mathsf{Spec}(\mathbb{Q})$ preserving rational equivalence of algebraic cycles.

Let k be an uncountable algebraically closed field of characteristic 0. Let T be an integral scheme over k, $\mathscr{X}_T$ be a scheme over T and $X_t = f_T^{-1}(t)$ be the fiber over $t \in T$ of the flat family $f_T : \mathscr{X}_T \to T$. Recall that a c-open subset of an integral scheme is the complement of a c-closed subset (Definition 1.3.11).

Lemma 4.2.1. *Given an integral base T over k there exits a natural c-open subset U_0 in T such that every $t \in U_0$ is scheme-theoretic isomorphic to the generic geometric point $\overline{\eta}$ point of T and given a flat family $f_T : \mathscr{X}_T \to T$ over T, the above scheme-theoretic isomorphism of points induce an isomorphism between the fiber X_t, for all closed points $t \in U_0$, and the geometric generic fiber $X_{\overline{\eta}}$, as schemes over $\mathsf{Spec}(\mathbb{Q})$, moreover these isomorphisms preserve rational equivalence of algebraic cycles (see [1, §5]).*

Proof. Assume that T is affine (as one can always cover the integral scheme T by open affine subschemes).

Step 1. We begin with the strategy of the construction of the c-open subset in T.

Recall that the transcendental degree $[k : \mathbb{Q}]$ of the uncountable algebraically closed field k over its primary subfield, i.e., over $\mathbb{Q}$, is infinity.

Since T is an integral affine scheme of finite type over k, then it is of the form

$$T = \mathsf{Spec}\Big(\frac{k[x_1, \dots, x_n]}{I(T)}\Big),$$

where $I(T) \subset k[x_1, \dots, x_n]$ is the ideal of T.

Let $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ be a set of generators of $I(T)$. As the polynomials f_i have a finite number of coefficients, attaching the coefficients of $f_1, \dots, f_m$ to $\mathbb{Q}$ we get an extension of $\mathbb{Q}$, say $\tilde{k}$, which is a countable subfield of k since $\mathbb{Q}$ is countable. Let k' be the algebraic closure of $\tilde{k}$, then it is a countable algebraically closed subfield of k.

Let T' be the affine integral scheme defined by the ideal $I(T')$ generated by

$$f_1, \dots, f_m \text{ in } k'[x_1, \dots, x_n].$$

Since

$$\frac{k[x_1, \dots, x_n]}{I(T)} = \frac{k'[x_1, \dots, x_n]}{I(T')} \otimes_{k'} k$$

we have

$$T = T' \times_{\mathsf{Spec}(k')} \mathsf{Spec}(k).$$

Denote by $k[T] = \frac{k[x_1,\dots,x_n]}{I(T)}$ (resp. $k[T'] = \frac{k'[x_1,\dots,x_n]}{I(T')}$) to the coordinate ring of T (resp. of T') and by $k(T)$ (resp. by $k(T')$) its function field.

Note that a closed subscheme Z' of T' is defined by an ideal $\mathfrak{a}$ in $k'[T'] = \frac{k'[x_1,\dots,x_n]}{I(T')}$, and let $i_{Z'} : Z' \hookrightarrow T'$ be the corresponding closed embedding. Since the field k' is countable and $\mathfrak{a}$ is finitely generated, there exist only countably many closed subschemes Z' in T'. For each Z' denote the complement by $U_{Z'} = T' \setminus \mathrm{im}(i_{Z'})$.

Let

$$Z = Z' \times_{\mathsf{Spec}(k')} \mathsf{Spec}(k), U_Z = U_{Z'} \times_{\mathsf{Spec}(k')} \mathsf{Spec}(k) \text{ and } i_Z : Z \hookrightarrow T$$

be the pullbacks of $Z', U_{Z'}$ and $i_{Z'}$ respectively, with respect to the extension k/k', then

$$U_Z = T \setminus \mathrm{im}(i_Z).$$

Let

$$U_0 = T \setminus \bigcup_{Z'} \mathrm{im}(i_Z) = \bigcap_{Z'} U_Z,$$

where the union is taken over closed subschemes Z', such that $\mathrm{im}(i_Z) \neq T$. So U_0 is the complement of the countable union of Zariski closed subsets, i.e., U_0 is c-open by construction.

Recall that a k-point t of a scheme T is a section of the structural morphism $h : T \to \mathsf{Spec}(k)$, that is, a morphism $f_t : \mathsf{Spec}(k) \to T$ such that $h \circ f_t = id_{\mathsf{Spec}(k)}$.

Step 2. Now we will see that there is an important isomorphism of fields related with each k-point of the c-open U_0 constructed above, that is, there is an isomorphism between the residue fields of the k-points of the c-open U_0 and the residue field of the geometric generic point of T. More precisely

Claim: Let $\overline{k(T)}$ be the algebraic closure of the field $k(T)$. For a k-point[1] t in U_0, one can construct a field isomorphism

$$e_t : \overline{k(T)} \xrightarrow{\sim} k$$

such that for $f \in k'[T']$ we have $e_t(f) = f(t)$.

Proof of the Claim. Let t be a k-point in U_0, that is, a morphism $f_t : \mathsf{Spec}(k) \to T$. Let

$$\pi : T = T' \times_{\mathsf{Spec}(k')} \mathsf{Spec}(k) \to T'$$

be the projection, since $t \in U_0$ by the construction of U_0 we have that $\pi(t) = \eta' \in \bigcap_{Z'} U_{Z'}$ where the intersection is taken over the closed subschemes Z' of T' such that $\mathrm{im}(i_{Z'}) \neq T'$. Therefore η' is the generic point of T', since the generic point of an integral scheme is unique. This is the same to say that there exists a morphism

$$h_t : \{t\} = \mathsf{Spec}(k) \to \mathsf{Spec}(k'(T')) = \eta',$$

such that the following diagram commutes

$$\begin{tikzcd} \{t\} = \mathsf{Spec}(k) \arrow[r, "f_t"] \arrow[d, "h_t"] & T \arrow[d, "\pi"] \\ \eta' = \mathsf{Spec}(k'(T')) \arrow[r, "g'"] & T' \end{tikzcd}$$

In terms of coordinate rings this means that there exist a homomorphism $\epsilon_t : k'(T') \to k$ such that the following diagram commutes

$$\begin{tikzcd} k & k[T] \arrow[l, "ev_t"'] \\ k'(T') \arrow[u, "\epsilon_t"] & k'[T'] \arrow[l] \arrow[u] \end{tikzcd}$$

Here k is considered as the residue field of the scheme T at t, $ev_t : k[T] \to k$ is the evaluation at t morphism, corresponding to the morphism f_t, and that ϵ_t is the homomorphism corresponding to the morphism h_t.

[1] Since in our case k is algebraically closed, k-points of T coincide with closed points of T. So this claim is true for all closed points of U_0.

Since $k'[T'] \to k[T]$ is injective, $k'[T'] \setminus \{0\}$ is a multiplicative system in $k[T]$. Furthermore we have

$$(k'[T'] \setminus \{0\})^{-1} k[T] = k[T] \otimes_{k'[T']} k'(T').$$

Hence there exists a unique universal morphism $\varepsilon_t : k[T] \otimes_{k'[T']} k'(T') \to k$ such that $\varepsilon_t \mid_{k[T]} = ev_t$ and $\varepsilon_t \mid_{k'(T')} = \epsilon_t$.

We now construct an embedding of $k(T) \hookrightarrow k$ whose restriction to $k'(T')$ is ϵ_t, that is, such that the following diagram commute

$$\begin{tikzcd} k(T) \arrow[r, hook] & k \\ k'(T') \arrow[u, hook] \arrow[ur, "\epsilon_t"] & \end{tikzcd}$$

Let $s = \dim(T') = Tr.\deg(k'(T)/k') =$ krull dimension of $k'[T']$. Here we denote by $Tr.\deg(k'(T)/k')$ to the transcendence degree of $k(T')$ over k', then by the Noether normalization lemma there exist s algebraically independent elements $x_1, \ldots, x_s$ in $k'[T']$ such that $k'[T']$ is a finitely generated module over the polynomial ring $k'[x_1, \ldots, x_s]$ and $k'(T')$ is algebraic over the field of fractions $k'(x_1, \ldots, x_s)$.

It follows that $k[T]$ is a finitely generated module over the polynomial ring $k[x_1, \ldots, x_s]$ and $k(T)$ is algebraic over the field of fractions $k(x_1, \ldots, x_s)$.

Let $b_i = ev_t(x_i)$ for $i = 1, \ldots, s$. Since $t \in U_0$ we have that $b_1, \ldots, b_s$ are algebraically independent over k'. Indeed, if $b_1, \ldots, b_s$ are algebraic dependent over k' there is a non-trivial polynomial f in s variables with coefficients in k' such that $f(b_1, \ldots, b_s) = 0$ or equivalently such that $f(ev_t(x_1), \ldots, ev_t(x_s)) = 0$, so we have a polynomial such that t is a zero of it, then $t \notin U_0$ which is a contradiction.

We can extend the set $b_1, \ldots, b_s$ to a transcendental basis B of k over k', so that $k = k'(B)$. As B have an infinite cardinality $B \setminus \{b_1, \ldots, b_s\}$ also have an infinite cardinality, choosing a bijection $B \xrightarrow{\sim} B \setminus \{b_1, \ldots, b_s\}$ we obtain the following field embedding

$$k = k'(B) \simeq k'(B \setminus \{b_1, \ldots, b_s\}) \subset k'(B)$$

over k' such that $b_1, \ldots, b_s$ is algebraically independent over its image. Then we get a field embedding

$$E_t : k(x_1, \ldots, x_s) \hookrightarrow k$$

sending x_i to b_i. Note that $E_t \mid_{k'(x_1,\ldots,x_s)} = \epsilon_t \mid_{k'(x_1,\ldots,x_s)}$.

Since

$$k(T) = k(x_1, \ldots, x_s) \otimes_{k'(x_1,\ldots,x_s)} k'(T'),$$

we get a uniquely defined embedding

$$k(T) \to k$$

as the composition $k(T) \to k(x_1, \dots, x_s) \hookrightarrow k$. The embedding $k(T) \to k$ can extend to an isomorphism

$$e_t : \overline{k(T)} \xrightarrow{\sim} k.$$

Finally, by the commutativity of the diagram

$$\begin{tikzcd} k & k[T] \arrow[l, "ev_t"'] \\ k'(T') \arrow[u, "\epsilon_t"] & k'[T'] \arrow[l] \arrow[u] \end{tikzcd}$$

if we take $f \in k'[T']$ we can identify it with its image via the inclusions $k'[T'] \to k[T]$ and $k'[T'] \to k'(T')$, then we have $ev_t(f) = \epsilon_t(f)$. since $e_t|_{k'(T')} = \epsilon_t$ we have $e_t(f) = f(t)$. $\square$

Step 3. Given a $f_T : \mathscr{X}_T \to T$ a smooth morphism of schemes over k, we now see that the above isomorphism of fields induces an isomorphism of the fibers of f_T.

Let $f_T : \mathscr{X}_T \to T$ be a smooth morphism of schemes over k. Extending, if necessary, the field k' used to construct the c-open U_0 we may assume that there exists a morphism of schemes $f'_{T'} : \mathscr{X}'_{T'} \to T'$ over the countable algebraically closed field k', such that f_T is the pullback of $f'_{T'}$ under the field extension k/k'.

Let

- $\eta' = \mathsf{Spec}(k'(T'))$ be the generic point of the affine scheme T', and let $X'_{\eta'}$ be the fibre of the family $f'_{T'} : \mathscr{X}'_{T'} \to T'$ over η',
- $\eta = \mathsf{Spec}(k(T))$ be the generic point of the affine scheme T, and let X_η be the fibre of the family $f_T : \mathscr{X}_T \to T$ over η,
- $\overline{\eta} = \mathsf{Spec}(\overline{k(T)})$ be the geometric generic point of the affine scheme T, and let $X_{\overline{\eta}}$ be the fibre of the family $f_T : \mathscr{X}_T \to T$ over $\overline{\eta}$.

The above isomorphism of fields

$$e_t : \overline{k(T)} \xrightarrow{\sim} k,$$

induces a scheme-theoretic isomorphism between the closed k-point $t \in U_0$ and the geometric generic point $\overline{\eta}$ of T

$$\mathsf{Spec}(e_t) : \{t\} = \mathsf{Spec}(k) \to \mathsf{Spec}(\overline{k(T)}) = \{\overline{\eta}\}$$

over η', since we have $h_t = \mathsf{Spec}(\epsilon_t) : \mathsf{Spec}(k) \to \eta'$ and $\mathsf{Spec}(\overline{k(T)}) \to \eta'$.

Pulling back the scheme-theoretic isomorphism $\mathsf{Spec}(e_t)$ onto the fibres of the family f_T we obtain the cartesian squares

$$\begin{tikzcd} X_t \arrow[r, "\kappa_t"] \arrow[d] & X_{\overline{\eta}} \arrow[d] \\ \mathsf{Spec}(k) = \{t\} \arrow[r, "\mathsf{Spec}(e_t)"] & \mathsf{Spec}(\overline{k(T)}) = \overline{\eta} \end{tikzcd}$$

and pulling back $\mathsf{Spec}(\epsilon_t)$ onto the fibers we obtain

$$\begin{tikzcd} X_t \arrow[r] \arrow[d] & X'_{\eta'} \arrow[d] \\ \{t\} = \mathsf{Spec}(k) \arrow[r, "\mathsf{Spec}(\epsilon_t)"] & \mathsf{Spec}(k'(T')) = \eta' \end{tikzcd}$$

similarly, we get $X_{\overline{\eta}} \to X'_{\eta'}$ by pulling back $\mathsf{Spec}(\overline{k(T)}) \to \mathsf{Spec}(k'(T'))$.

Note that the morphism κ_t induced by $\mathsf{Spec}(e_t) = h_t$ is an isomorphism of schemes over $X'_{\eta'}$.

Step 4. Next we describe the isomorphism between fibres X_t and $X_{t'}$ with $t, t' \in U_0$.

Let t' be another closed point of U_0, then we also have the isomorphism

$$e_{t'} : \overline{k(T)} \xrightarrow{\sim} k(t') = k,$$

then

$$\sigma_{tt'} : k(t) = k \xrightarrow{e_t^{-1}} \overline{k(T)} \xrightarrow{e_{t'}} k = k(t')$$

is an automorphism of k.

Let $(X_t)_{\sigma_{tt'}} = X_t \times_{\mathsf{Spec}(k)} \mathsf{Spec}(k)$ with respect to the automorphism of schemes $\mathsf{Spec}(\sigma_{tt'}) : \mathsf{Spec}(k) \to \mathsf{Spec}(k)$ induced by $\sigma_{tt'}$, and let

$$w_{\sigma_{tt'}} : (X_t)_{\sigma_{tt'}} \xrightarrow{\sim} X_t$$

be the corresponding isomorphism on schemes over $\mathsf{Spec}(k^{\sigma_{tt'}})$, where $k^{\sigma_{tt'}}$ is a subfield of $\sigma_{tt'}$-invariants in k.

Since $k' \subset k^{\sigma_{tt'}} \subset k$, the projection $\mathscr{X}_T \to \mathscr{X}'_{T'}$ factorises through

$$(\mathscr{X}'_{T'})_{k^{\sigma_{tt'}}} = \mathscr{X}'_{T'} \times_{\mathsf{Spec}(k')} \mathsf{Spec}(k^{\sigma_{tt'}}),$$

i.e., we have the following commutative diagram

$$\begin{tikzcd} \mathscr{X}_T \arrow[r] \arrow[dr] & \mathscr{X}'_{T'} \\ & (\mathscr{X}'_{T'})_{k^{\sigma_{tt'}}} \arrow[u] \end{tikzcd}$$

So, we can consider the fiber X_t as a scheme over $(\mathscr{X}'_{T'})_{k^{\sigma_{tt'}}}$ just by composing the inclusion of $X_t \hookrightarrow \mathscr{X}_T$ with the morphism $\mathscr{X}_T \to (\mathscr{X}'_{T'})_{k^{\sigma_{tt'}}}$.

Recall that $\sigma_{tt'} : k(t) = k \to k = k(t')$ and let

$$\kappa_{tt'} : X_{t'} \xrightarrow{\kappa_{t'}} X_{\overline{\eta}} \xrightarrow{\kappa_t^{-1}} X_t$$

be the induced isomorphism of the fibres as schemes over $\mathsf{Spec}(k^{\sigma_{tt'}})$. It follows that

- $(X_t)_{\sigma_{tt'}} = X_{t'}$,
- the isomorphism $w_{\sigma_{tt'}} : X_{t'} \xrightarrow{\sim} X_t$ is over $(\mathscr{X}'_{T'})_{k^{\sigma_{tt'}}}$, and
- $w_{\sigma_{tt'}} = \kappa_{tt'}$.

Step 5. Finally, we see that the isomorphisms of fibres for all closed points of U_0 preserve rational equivalence. More precisely:

Claim: the scheme-theoretic isomorphisms κ_t, for $t \in U_0$, preserve rational equivalence of algebraic cycles.

For the proof of this claim see [1, Lemma 19]. $\square$

Facts on a family of hyperplane sections of a surface

For any integral scheme T over $\mathbb{C}$ and for any morphism of schemes $T \to (\mathbb{P}^d)^*$, let

$$f_T : \mathscr{C}_T \to T$$

be the family of hyperplane sections of S parametrized by T,

$$g_T : \mathcal{S}_T \to T$$

the trivial family, that is, the family such that each fiber over T is isomorphic to S, and

$$\begin{tikzcd} \mathscr{C}_T \arrow[rr, "r_T"] \arrow[dr, "f_T"'] & & \mathcal{S}_T \arrow[dl, "g_T"] \\ & T & \end{tikzcd}$$

the closed embedding of schemes over T. Then we also have closed embeddings $r_t : C_t \hookrightarrow S$ and $r_{\overline{\eta}} : C_{\overline{\eta}} \to S_{\overline{\eta}}$ over $t = \mathsf{Spec}(\mathbb{C})$ and $\overline{\eta}$ respectively.

By Lemma 4.2.1 there exits a natural c-open subset U_0 in T such that the residue field of any closed point in U_0 is isomorphic to the residue field of the geometric generic point of T, since this isomorphism of fields induce a scheme-theoretic isomorphism between points, this c-open subset U_0 is such that any closed point $t \in U_0$ is scheme-theoretic isomorphic to the geometric generic point $\overline{\eta}$ of T.

Extending appropriately, if necessary, the countably algebraically closed field k', used to construct U_0, we may assume that there exists morphisms of schemes $f'_{T'}$, $g'_{T'}$ and $r'_{T'}$ over k' with $f'_{T'} = g'_{T'} \circ r'_{T'}$, that is, with

$$\begin{tikzcd} \mathscr{C}'_{T'} \arrow[rr, "r'_{T'}"] \arrow[dr, "f'_{T'}"'] & & \mathcal{S}'_{T'} \arrow[dl, "g'_{T'}"] \\ & T' & \end{tikzcd}$$

and such that f_T, g_T and r_T are the pullback of $f'_{T'}$, $g'_{T'}$ and $r'_{T'}$ respectively under the field extension $\mathbb{C}/k'$. Then the scheme-theoretic isomorphism between the points $t \in U_0$ and the geometric generic point $\overline{\eta}$ of T induce isomorphisms $\kappa_t^{f_T} : C_t \to C_{\overline{\eta}}$ (resp. $\kappa_t^{g_T} : S_t \to S_{\overline{\eta}}$) between the fiber C_t (resp. S_t) over t and the geometric generic fiber $C_{\overline{\eta}}$ (resp. $S_{\overline{\eta}}$) over $\overline{\eta}$ of the family f_T (resp. g_T) for every $t \in U_0$, as schemes over $\mathsf{Spec}(\mathbb{Q})$, and for any two points t and t' in U_0 one has the isomorphisms $\kappa_{tt'}^{f_T} : C_t \to C_{t'}$ (resp. $\kappa_{tt'}^{g_T} : S_t \to S_{t'}$). Moreover, for any closed point $t \in U_0$, the following diagram

$$\begin{tikzcd} C_t \arrow[r, "r_t"] \arrow[d, "\kappa_t^{f_T}"] & S_t \arrow[d, "\kappa_t^{g_T}"] \\ C_{\overline{\eta}} \arrow[r, "r_{\overline{\eta}}"] & S_{\overline{\eta}} \end{tikzcd} \tag{4.3}$$

commutes, where r_t and $r_{\overline{\eta}}$ are the morphisms on fibres induced by r_T. Then the isomorphisms $\kappa_{tt'}^{f_T} = (\kappa_t^{f_T})^{-1} \circ \kappa_{t'}^{f_T}$ (resp. $\kappa_{tt'}^{g_T} = (\kappa_t^{g_T})^{-1} \circ \kappa_{t'}^{g_T}$) commute with closed embeddings r_t and $r_{t'}$ for any two closed points t, t' in U_0. Removing more Zariski closed subset from U_0 if necessary we may assume that the fibres of the families f_T and g_T over the points on U_0 are smooth, that is, we can assume that $U_0 \subset U$.

For every closed point $t \in U_0$, let

$$alb_{C_t} : \mathsf{CH}_0(C_t)_{\deg=0} \xrightarrow{\sim} J_t$$

be the corresponding isomorphisms given by fact 1 (see Lemma 2.4.12), and denote by

$$alb_{C_{\overline{\eta}}} : \mathsf{CH}_0(C_{\overline{\eta}})_{\deg=0} \xrightarrow{\sim} J_{\overline{\eta}}$$

the isomorphism for the geometric generic fiber (see Remark 2.4.13).

By the step 5 of Lemma 4.2.1, for any $t \in U_0$ the scheme-theoretic isomorphisms

$$\kappa_t^{f_T} : C_t \to C_{\overline{\eta}}$$

of the family f_T preserve rational equivalence, then they induce the push-forward isomorphisms of Chow groups

$$\kappa_{t*}^{f_T} : \mathsf{CH}_0(C_t)_{\deg=0} \to \mathsf{CH}_0(C_{\overline{\eta}})_{\deg=0},$$

then we get $l_t : J_t \to J_{\overline{\eta}}$ as the composition given by the commutative diagram

$$\begin{CD}
\mathsf{CH}_0(C_t)_{\deg=0} @>{\kappa_{t*}^{f_T}}>> \mathsf{CH}_0(C_{\overline{\eta}})_{\deg=0}\\
@A{alb_{C_t}^{-1}}AA @VV{alb_{C_{\overline{\eta}}}}V\\
J_t @>{l_t}>> J_{\overline{\eta}}
\end{CD}$$

Now consider the following commutative diagram

$$\begin{CD}
\mathsf{Sym}^{d,d}(C_t) @>{\mathsf{Sym}^{d,d}(\kappa_t^{f_T})}>> \mathsf{Sym}^{d,d}(C_{\overline{\eta}})\\
@V{\theta_d^{C_t}}VV @VV{\theta_d^{C_{\overline{\eta}}}}V\\
\mathsf{CH}_0(C_t)_{\deg=0} @>{\kappa_{t*}^{f_T}}>> \mathsf{CH}_0(C_{\overline{\eta}})_{\deg=0}\\
@V{alb_{C_t}}VV @VV{alb_{C_{\overline{\eta}}}}V\\
J_t @>{l_t}>> J_{\overline{\eta}}
\end{CD}$$

Since $alb_{C_t} \circ \theta_d^{C_t}$ is a regular morphism of schemes over $\mathbb{C}$ and $alb_{C_{\overline{\eta}}} \circ \theta_d^{C_{\overline{\eta}}}$ is a regular morphism of schemes over $\overline{\mathbb{C}(T)}$ the algebraic closure of the function field of T (see Lemma 1.3.21 and Lemma 1.3.23) and the morphism $\mathsf{Sym}^{d,d}(\kappa_t^{f_T})$ is a regular morphism over $\mathbb{Q}$, it follows that the homomorphism $l_t : J_t \to J_{\overline{\eta}}$ is a regular morphism of schemes over $\mathbb{Q}$.

Similarly, by step 5 of Lemma 4.2.1, for any $t \in U_0$ the scheme-theoretic isomorphisms

$$\kappa_t^{g_T} : S_t \to S_{\overline{\eta}}$$

on the fibers of the family g_T preserve rational equivalence, then they induce the push-forward isomorphisms of abelian groups

$$\kappa_{t*}^{g_T} : \mathsf{CH}_0(S_t)_{\deg=0} \to \mathsf{CH}_0(S_{\overline{\eta}})_{\deg=0}.$$

and from the commutative diagram (4.3) one obtains the commutative diagram in Chow groups

$$\begin{CD}
\mathsf{CH}_0(C_t)_{\deg=0} @>{r_{t*}}>> \mathsf{CH}_0(S_t)_{\deg=0}\\
@V{\kappa_{t*}^{f_T}}VV @VV{\kappa_{t*}^{g_T}}V\\
\mathsf{CH}_0(C_{\overline{\eta}})_{\deg=0} @>{r_{\overline{\eta}*}}>> \mathsf{CH}_0(S_{\overline{\eta}})_{\deg=0}
\end{CD} \tag{4.4}$$

For every closed $t \in U_0$, let A_t and B_t be the abelian subvarieties of J_t obtained in the proof of item a) and let $A_{\overline{\eta}}$ and $B_{\overline{\eta}}$ be the abelian subvarieties of $J_{\overline{\eta}}$ which can be obtained in a similar way to the proof of item a) corresponding to the closed embedding $r_{\overline{\eta}} : C_{\overline{\eta}} \to S_{\overline{\eta}}$ (see Remark 4.1.4).

Lemma 4.2.2. *For any closed point $t \in U_0$, $l_t(B_t) = B_{\overline{\eta}}$ and $l_t(A_t) = A_{\overline{\eta}}$.*

Proof. To prove $l_t(A_t) = A_{\overline{\eta}}$ recall that by item a) we have

$$G_t = \bigcup_{x \in \Xi_t} (x + A_t) \text{ and } G_{\overline{\eta}} = \bigcup_{x \in \Xi_{\overline{\eta}}} (x + A_{\overline{\eta}}).$$

By definition $l_t = alb_{C_{\overline{\eta}}} \circ \kappa_{t*}^{f_T} \circ alb_{C_t}^{-1}$ then

$$l_t(G_t) = alb_{C_{\overline{\eta}}} \circ \kappa_{t*}^{f_T} \circ alb_{C_t}^{-1}(G_t)$$

equivalently we have

$$l_t(G_t) = alb_{C_{\overline{\eta}}} \circ \kappa_{t*}^{f_T}(G_t)$$

via the identification $alb_{C_t}^{-1}$. By the commutative diagram (4.4) we have $\kappa_{t*}^{f_T}(G_t) = G_{\overline{\eta}}$, then

$$l_t(G_t) = G_{\overline{\eta}} \tag{4.5}$$

via the isomorphism $alb_{C_{\overline{\eta}}}$.

On the other hand,

$$l_t(G_t) = l_t(\bigcup_{x\in\Xi_t}(x + A_t)) = \bigcup_{x\in\Xi_t}(l_t(x) + l_t(A_t)) \tag{4.6}$$

By equations (4.5) and (4.6) we have

$$\bigcup_{x\in\Xi_t}(l_t(x) + l_t(A_t)) = \bigcup_{x\in\Xi_{\overline{\eta}}}(x + A_{\overline{\eta}}).$$

Note that $l_t(A_t)$ is Zariski closed in $J_{\overline{\eta}}$ since the group isomorphism l_t are regular morphisms of schemes over $\mathsf{Spec}(\mathbb{Q})$. Since $l_t(A_t)$ is a subgroup of in $J_{\overline{\eta}}$, it is an abelian subvariety in $J_{\overline{\eta}}$.

As the right and left terms of the above equality are irredundant decomposition of $G_{\overline{\eta}}$ by the uniqueness of it (see Lemma 1.3.9) and by the fact that the irredundant decomposition of $G_{\overline{\eta}}$ must contain a unique irreducible component passing through 0 (see Lemma 1.3.10) we have that $l(A_t) = A_{\overline{\eta}}$. □

Remark 4.2.3. The above Lemma 4.2.2 tells us that one can study the varieties A_t in a family either working at the geometric generic point or at a very general closed point on the base scheme.

Facts on Lefschetz pencils of hyperplanes sections of a surface intersecting U_0

Now, choose $L \cong \mathbb{P}^1$ be a Lefschetz pencil of hyperplanes for the surface S (see Definition 3.1.6 and Proposition 3.1.10) such that $L \cap U_0 \neq \emptyset$.

Lemma 4.2.4. *Let $t \in L \cap U_0$, let A_t be the abelian subvariety of $B_t \subset J_t$ obtained in the proof of item a). Then either $A_t = 0$ or $A_t = B_t$.*

Proof. Since $L \cong \mathbb{P}^1$ be a Lefschetz pencil of hyperplanes for the surface S passing through t (if we think of this Lefschetz pencil as the family of hyperplane sections

$(C_t)_{t\in L}$ parametrized by L this means that C_t corresponding to this t is a member of this family), then it gives rise to a morphism

$$f_L : \mathscr{C}_L \to L,$$

where $\mathscr{C}_L$ is smooth because it can be identified with the blowing up

$$\tilde{S} = \{(x,t) \in S \times L : x \in C_t = S \cap H_t\}$$

of S at the base locus A_L of the pencil, and $f_L = \mathsf{pr}_2|_{\tilde{S}} : \tilde{S} \to L$. Moreover, each hyperplane section C_t of S parametrized by points of L can be naturally identified with the fibre $f_L^{-1}(t) \subset \tilde{S}$, so each fibre C_t of f_L has at most one ordinary double point as singularity.

Let $\{0_1, \ldots, 0_M\}$ be the critical values of the Lefschetz pencil L and $V = L - \{0_1, \ldots, 0_M\}$, then we have the monodromy action

$$\rho_V : \pi_1(V,t) \to \mathsf{Aut}(H^1(C_t, \mathbb{Q}))_{\mathsf{van}},$$

associated to the local system $R^1 f_L \mathbb{Z}|_V$ see [12, §5]. Now we claim that the local monodromy representation ρ_V is irreducible.

Proof of the claim. Indeed, recall that $\{0_1, \ldots, 0_M\}$ are the critical values of the Lefschetz pencil $L \cong \mathbb{P}^1$. For each 0_i with $i = 1, \ldots, M$, consider the small disk $D_i \subset L$ centered at 0_i, $t_i \in D_i^*$ and γ_i the path joining t to t_i. Let $\delta_i' \in H_1(C_{t_i}, \mathbb{Q}) = H^1(C_{t_i}, \mathbb{Q})$ be the vanishing cycle (the homology class of the vanishing sphere $S^1_{t_i} \subset C_{t_i}$) which is well defined up to sign as a generator of $\mathsf{Ker}(H^1(C_{t_i}, \mathbb{Q}) \to H^1(C_{\Delta_i}, \mathbb{Q}))$ and recall that by trivialising the fibration $f_L|_{\gamma_i}$ over γ_i we can construct a diffeomorphism $C_{t_i} \cong C_t$, so we have a vanishing cycle $\delta_i \in H^1(C_t, \mathbb{Q})$ which is image of δ_i' via the diffeomorphism. By Lemma 3.2.19 the vanishing cohomology $H^1(C_t, \mathbb{Q})_{\mathsf{van}}$ is generated by these vanishing cycles δ_i, $i = 1, \ldots, M$, of the Lefschetz pencil L.

On the other hand, let $\tilde{\gamma}_i$ be the loop in V based at t such that $\tilde{\gamma}_i$ is equal to γ_i until t_i winds around the disk D_i once in the positive direction, and then returns to t via γ_i^{-1}. Recall that these loops $\tilde{\gamma}_i, i = 1, \ldots, M$, generate $\pi_1(V,t)$ and note that the image of the loops $\tilde{\gamma}_i$ via ρ_V are elements in $\mathsf{Aut}(H^1(C_t, \mathbb{Q})_{\mathsf{van}})$, that is,

$$\rho_V(\tilde{\gamma}_i) : H^1(C_t, \mathbb{Q})_{\mathsf{van}} \to H^1(C_t, \mathbb{Q})_{\mathsf{van}}$$

are automorphisms.

Let $F \subset H^1(C_t, \mathbb{Q})_{\mathsf{van}}$ be a nontrivial vector subspace which is stable under the monodromy action ρ_V. We must prove that $F = H^1(C_t, \mathbb{Q})_{\mathsf{van}}$ or $F = \{0\}$.

Let $0 \neq \alpha \in F$. Since by Proposition 3.2.21, $\langle , \rangle$ is non-degenerate on $H^1(C_t, \mathbb{Q})_{\mathsf{van}}$ there exists $i \in \{1, \ldots, M\}$ such that $\langle \alpha, \delta_i \rangle \neq 0$.

By the Picard-Lefschetz formula (Theorem 3.4.2) for $\alpha \in F \subset H^1(C_t, \mathbb{Q})_{\mathsf{van}}$ one has $\rho_V(\tilde{\gamma}_i)(\alpha) = \alpha \pm \langle \alpha, \delta_i \rangle \delta_i$ or equivalently, $\rho_V(\tilde{\gamma}_i)(\alpha) - \alpha = \pm \langle \alpha, \delta_i \rangle \delta_i$.

Since, by assumption, F is a vector subspace of $H^1(C_t, \mathbb{Q})_{\mathsf{van}}$ which is stable under the monodromy action ρ_V (so $\rho_V(\tilde{\gamma}_i)(\alpha) \in F$) we have $\rho_V(\tilde{\gamma}_i)(\alpha) - \alpha = \pm\langle \alpha, \delta_i \rangle \delta_i \in F$. Then $\delta_i \in F$.

But, Corollary 3.4.11 shows that all the vanishing cycles are conjugate under the monodromy action, so F, which is stable under the monodromy action, must contain all the vanishing cycles. Thus $F = H^1(C_t, \mathbb{Q})_{\mathsf{van}}$. □

On the other hand, by Lemma 2.3.6 (fact 3) we have $H^1(C_t, \mathbb{Z}) \overset{w_{t*}}{\simeq} H^1(J_t, \mathbb{Z})$ is an isomorphism. Let

$$H^1(C_t, \mathbb{Q}) = H^1(C_t, \mathbb{Z}) \otimes \mathbb{Q} \overset{(w_{t*})_{\mathbb{Q}}}{\simeq} H^1(J_t, \mathbb{Z}) \otimes \mathbb{Q} = H^1(J_t, \mathbb{Q})$$

be the isomorphism induced by w_{t*}, we get this because in particular C_t is compact (see [29, §7.1.1]).

Let $L_t = (w_{t*})_{\mathbb{Q}}^{-1}(H^1(A_t, \mathbb{Q}))$ be the (pre)image in $H^1(C_t, \mathbb{Q})$ of $H^1(A_t, \mathbb{Q}) \subset H^1(J_t, \mathbb{Q})$ under the isomorphism $(w_{t*})_{\mathbb{Q}}^{-1}$. Then L_t is a $\mathbb{Q}$-vector subspace in $H^1(C_t, \mathbb{Q})$.

Recall also that $H^1(B_t, \mathbb{Z}) \overset{w_{t*}}{\simeq} H^1(C_t, \mathbb{Z})_{\mathsf{van}}$ (see final part of the proof of item a)), then it follows that

$$H^1(B_t, \mathbb{Q}) \overset{(w_{t*})_{\mathbb{Q}}}{\simeq} H^1(C_t, \mathbb{Q})_{\mathsf{van}}.$$

Since in item a) we proved that $H^1(A_t, \mathbb{Z}) \subset H^1(B_t, \mathbb{Z})$ we get

$$H^1(A_t, \mathbb{Q}) \subset H^1(B_t, \mathbb{Q}),$$

this implies that $L_t \subset H^1(C_t, \mathbb{Q})_{\mathsf{van}}$ in $H^1(C_t, \mathbb{Q})$ via the isomorphism $(w_{t*})_{\mathbb{Q}}$. Moreover, L_t is a vector subspace in $H^1(C_t, \mathbb{Q})_{\mathsf{van}}$ which has a Hodge structure on it since it corresponds to the abelian subvariety A_t of J_t, then it is invariant under the monodromy representation ρ_V (see Proposition 3.3.15).

Then, since the monodromy action

$$\rho_V : \pi_1(V, t) \to \mathsf{Aut}(H^1(C_t, \mathbb{Q})_{\mathsf{van}})$$

on $H^1(C_t, \mathbb{Q})_{\mathsf{van}}$ is irreducible, either $L_t \overset{(w_{t*})_{\mathbb{Q}}}{\simeq} H^1(A_t, \mathbb{Q}) = 0$ and then $A_t = 0$, or $L_t \overset{(w_{t*})_{\mathbb{Q}}}{\simeq} H^1(A_t, \mathbb{Q}) = H^1(C_t, \mathbb{Q})_{\mathsf{van}} \overset{(w_{t*})_{\mathbb{Q}}}{\simeq} H^1(B_t, \mathbb{Q})$ and then $A_t = B_t$. □

Finally, we next prove the item b) of the main result of this thesis.

Proof of item b) of Theorem 4.1.1. Let

$$f : \mathscr{C} \to \mathbb{P}^{d*}$$

be the universal hyperplane section of S, i.e., the family of hyperplane sections of S parametrized by $\mathbb{P}^{d*}$ (see Example 3.4.6).

Let

$$g : \mathcal{S} = S \times \mathbb{P}^{d*} \to \mathbb{P}^{d*}$$

be the trivial family parametrised by $\mathbb{P}^{d*}$, i.e., the family such that each fiber over $\mathbb{P}^{d*}$ is isomorphic to S.

Let

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "r"] \arrow[dr, "f"'] & & \mathcal{S} \arrow[dl, "g"] \\ & \mathbb{P}^{d*} & \end{tikzcd}$$

be the closed embedding of schemes over $\mathbb{P}^{d*}$.

By Lemma 4.2.1 there exits a natural c-open subset U_0 in $\mathbb{P}^{d*}$ such that the residue field of any closed point in U_0 is isomorphic to the residue field of the geometric generic point of $\mathbb{P}^{d*}$, since this isomorphism of fields induce a scheme-theoretic isomorphism between points, this c-open U_0 is such that any closed point $t \in U_0$ is scheme-theoretic isomorphic to the geometric generic point $\overline{\eta}$ of $\mathbb{P}^{d*}$.

Extending appropriately the countably algebraically closed field $k' \subset \mathbb{C}$, used to construct U_0, we may assume that there exists morphisms of schemes f', g' and r' over k' with $f' = g' \circ r'$ and such that f, g and r are the pullback of f', g' and r' respectively under the field extension $\mathbb{C}/k'$. Then the scheme-theoretic isomorphism between the points $t \in U_0$ and the geometric generic point $\overline{\eta}$ of $\mathbb{P}^{d*}$ induce isomorphisms $\kappa_t^f : C_t \to C_{\overline{\eta}}$ (resp. $\kappa_t^g : S_t \to S_{\overline{\eta}}$) between the fiber C_t (resp. S_t) over t and the geometric generic fiber $C_{\overline{\eta}}$ (resp. $S_{\overline{\eta}}$) over $\overline{\eta}$ of the family f (resp. g) for every $t \in U_0$, as schemes over $\mathsf{Spec}(\mathbb{Q})$, and for any two points t and t' in U_0 one has isomorphisms $\kappa_{tt'}^f : C_t \to C_{t'}$ (resp. $\kappa_{tt'}^g : S_t \to S_{t'}$). Moreover, for any closed point $t \in U_0$, the following diagram

$$\begin{tikzcd} C_t \arrow[r, "r_t"] \arrow[d, "\kappa_t^f"] & S_t \arrow[d, "\kappa_t^g"] \\ C_{\overline{\eta}} \arrow[r, "r_{\overline{\eta}}"] & S_{\overline{\eta}} \end{tikzcd}$$

commutes, where r_t and $r_{\overline{\eta}}$ are the morphisms on fibres induced by r. Then the isomorphisms $\kappa_{tt'}^f = (\kappa_t^f)^{-1} \circ \kappa_{t'}^f$ (resp. $\kappa_{tt'}^g = (\kappa_t^g)^{-1} \circ \kappa_{t'}^g$) commute with closed embeddings r_t and $r_{t'}$ for any two closed points t, t' in U_0. Removing more Zariski closed subset from U_0 if necessary we may assume that the fibres of the families f and g over the points on U_0 are smooth, that is, we can assume that $U_0 \subset U$.

Recall that for every closed point $t \in \mathbb{P}^{d*}$ we denote by H_t the corresponding hyperplane in $\mathbb{P}^d$.

Let $\Omega \subset \mathbb{P}^{d*}$ be a Zariski closed subset in $\mathbb{P}^{d*}$ such that for every point in $t \in \mathbb{P}^{d*} - \Omega$ the corresponding hyperplane H_t does not contain S and $H_t \cap S = C_t$ is either smooth or contains at most one singular point which is a double point.

Let $G(1,\mathbb{P}^{d*})$ be the Grassmannian of lines in $\mathbb{P}^{d*}$. There exists $W \subset G(1,\mathbb{P}^{d*})$ a Zariski open subset of $G(1,\mathbb{P}^{d*})$ such that for every line $L \in W$ we have $L \cap \Omega = \emptyset$ and its corresponding codimension 2 linear subspace A_L in $\mathbb{P}^d$ intersects S transversally. In other words, any line $L \in W$ gives rise to a Lefschetz pencil for S (see Corollary 3.1.12).

Let $Z = \mathbb{P}^{d*} - U_0$ be the complement of the c-open U_0 subset of $\mathbb{P}^{d*}$, then Z is c-closed. It follows that the condition for a line $L \in G(1,\mathbb{P}^{d*})$ to be not a subset in Z is c-open. This means that there exists a c-open $A \subset G(1,\mathbb{P}^{d*})$ such that for $L \in A$ we have $L \not\subset Z$. It follows that $A \cap W \neq \emptyset$, so we can choose a line $L \subset \mathbb{P}^{d*}$ such that gives rise to a Lefschetz pencil $f_L : \mathscr{C}_L \to L$ and $L \cap U_0 \neq \emptyset$.

Let $t_0 \in L \cap U_0$, then by Lemma 4.2.4 $A_{t_0} = 0$ or $A_{t_0} = B_{t_0}$.

Suppose that $A_{t_0} = 0$. Applying the Lemma 4.2.2 to the case $T = \mathbb{P}^{d*}$, we obtain $A_{\overline{\eta}} = 0$ because t_0 and $\overline{\eta}$ are isomorphic since $t_0 \in U_0$. Then applying the same Lemma 4.2.2 we have $A_t = 0$ for each closed point $t \in U_0$.

Suppose that $A_{t_0} = B_{t_0}$. Applying the Lemma 4.2.2 to the case $T = \mathbb{P}^{d*}$, we obtain $A_{\overline{\eta}} = B_{\overline{\eta}}$ because t_0 and $\overline{\eta}$ are isomorphic since $t_0 \in U_0$. Then applying the same Lemma 4.2.2 we have $A_t = B_t$ for each closed point $t \in U_0$.

□

4.3 On the proof of item c)

Proof of item c) of Theorem 4.1.1. Recall that $\Delta_S \subset \Sigma$ is the irreducible variety of $\Sigma = \mathbb{P}^{d*}$ parametrizing singular hyperplane sections of S and that $U = \Sigma \setminus \Delta_S$ is the open subset of $\Sigma = \mathbb{P}^{d*}$ parametrizing smooth hyperplane sections of S.

Let $f_U : \mathscr{C}_U \to U$ be the smooth universal hyperplane section of S, that is, the family of smooth hyperplane sections of S parametrized by U (see Definition 3.4.7).

Let $g_U : \mathcal{S}_U = S \times U \to U$ be the trivial family parametrized by U, that is, the family such that each fiber over U is isomorphic to S.

Let

$$\begin{tikzcd} \mathscr{C}_U \arrow[rr, "r_U"] \arrow[dr, "f_U"'] & & \mathcal{S}_U \arrow[dl, "g_U"] \\ & U & \end{tikzcd}$$

be the inclusion of fibrations over U.

Fix any (closed) point $t \in U = \Sigma \setminus \Delta$, then the fiber of the inclusion r_U over t is the closed embedding $r_t : C_t \hookrightarrow S_t \cong S$ of the smooth (hence connected, see [11]) curve C_t into the surface S over $\mathbb{C}$.

Recall that $\Delta_S^0 \subset \Delta_S$ is the open dense subset of Δ_S parametrizing hyperplanes in $\mathbb{P}^d$ such that the corresponding hyperplane sections of S has exactly one ordinary double point as singularity (see Definition 3.1.8).

Suppose in addition that $\dim(\Delta_S) = d-1$, i.e., the discriminant variety Δ_S of S is a hypersurface (e.g. S is a non-linear surface, see Proposition 3.1.13) by Proposition 3.1.10 any Lefschetz pencil of hyperplane sections of S meets the discriminant hypersurface Δ_S transversely in the open dense subset $\Delta_S^0 \subset \Delta_S$. So, in particular any Lefschetz pencil of hyperplane sections of S passing through t meets the discriminant hypersurface Δ_S transversely in the open dense subset Δ_S^0, then from Remark 3.4.8 and Zariski Theorem 3.4.9 we can conclude that there is a monodromy representation

$$\rho_{f_U} : \pi_1(U,t) \to \mathsf{Aut}(H^1(C_t,\mathbb{Q})),$$

associated to the local system $R^1 f_{U*}\mathbb{Q}$ corresponding to the fibration f_U (see Proposition 3.3.13).

Now recall that the above inclusion of fibrations r_U gives a morphism of local systems

$$r_{U*} : R^1 f_{U*}\mathbb{Q} \to R^3 g_{U*}\mathbb{Q}$$

whose stalk at the point t is

$$r_{t*} : H^1(C_t,\mathbb{Q}) \to H^3(S,\mathbb{Q}),$$

the Gysin homomorphism in cohomology groups. Thus we have a local subsystem in vanishing cohomology, that is, we have a local subsystem

$$\mathsf{Ker}(r_{U*} : R^1 f_{U*}\mathbb{Q} \to R^3 g_{U*}\mathbb{Q})$$

whose stalk at point $t \in U$ is $H^1(C_t,\mathbb{Q})_{\mathsf{van}} = \mathsf{Ker}(r_{t*} : H^1(C_t,\mathbb{Q}) \to H^3(S,\mathbb{Q}))$. By Proposition 3.4.12 we have that the monodromy representation ρ_{f_U} preserves the stalk of this local subsystem, i.e., leaves $H^1(C_t,\mathbb{Q})_{\mathsf{van}}$ stable. Then we have a monodromy representation on vanishing cohomology $H^1(C_t,\mathbb{Q})_{\mathsf{van}}$, that is, we have

$$\rho_{f_U} : \pi_1(U,t) \to \mathsf{Aut}(H^1(C_t,\mathbb{Q})_{\mathsf{van}}).$$

By Zariski's theorem (Theorem 3.4.9) the monodromy representation ρ_{f_U} can be computed by restricting to a Lefschetz pencil passing through t. So, let $L \cong \mathbb{P}^1$ be a Lefschetz pencil passing through t (if we think of this Lefschetz pencil as the family of hyperplane sections $(C_t)_{t\in L}$ parametrized by L this means that C_t corresponding to this t is a member of this family), then it gives rise to a morphism

$$f_L : \mathscr{C}_L \to L$$

where $\mathscr{C}_L$ is smooth because it can be identified with the blowing up

$$\tilde{S} = \{(x,t) \in S \times L : x \in C_t = S \cap H_t\}$$

of S at the base locus A_L of the pencil, and $f_L = \mathsf{pr}_2|_{\tilde{S}} : \tilde{S} \to L$. Moreover, each hyperplane section C_t of S parametrized by points of L can be naturally identified with the fibre $f_L^{-1}(t) \subset \tilde{S}$, so each fibre C_t of f_L has at most one ordinary double point as singularity. This Lefschetz pencil L through t exists because by Proposition 3.1.10 a restatement of the claim is that for t in $U = \mathbb{P}^{d*} \setminus \Delta_S$ there is a line in $\mathbb{P}^{d*}$ through t transverse to the discriminant variety Δ_S of S in the open dense subset Δ_S^0 and this is clear, see also Remark 3.2.14.

Let $V = L - L \cap \Delta_S$, then we have the monodromy action

$$\rho_{f_V} := \rho_{f_U} |_V \colon \pi_1(V, t) \to \mathsf{Aut}(H^1(C_t, \mathbb{Q}))$$

obtained by restringing us to the Lefschetz pencil $f_L : \mathscr{C}_L \to L$.

Now we claim that the local monodromy representation

$$\rho_{f_V} : \pi_1(V, t) \to \mathsf{Aut}(H^1(C_t, \mathbb{Q})_{\mathsf{van}})$$

is irreducible, that is, every vector subspace stable under ρ_{f_V} is equal to $\{0\}$ or $H^1(C_t, \mathbb{Q}))_{\mathsf{van}}$.

Proof of the claim. Indeed, Let $L \cap \Delta_S = \{0_1, \ldots, 0_M\}$ be the critical values of the Lefschetz pencil $L \cong \mathbb{P}^1$. For each 0_i with $i = 1, \ldots, M$, consider the small disk $D_i \subset L$ centered at 0_i, $t_i \in D_i^*$ and γ_i the path joining t to t_i. Let $\delta_i' \in H_1(C_{t_i}, \mathbb{Q}) = H^1(C_{t_i}, \mathbb{Q})$ be the vanishing cycle (the homology class of the vanishing sphere $S^1_{t_i} \subset C_{t_i}$) which is well defined up to sign as a generator of $\mathsf{Ker}(H^1(C_{t_i}, \mathbb{Q}) \to H^1(C_{\Delta_i}, \mathbb{Q}))$ and since that by trivialising the fibration $f_L|_{\gamma_i}$ over γ_i we can construct a diffeomorphism $C_{t_i} \cong C_t$, so we have a vanishing cycle $\delta_i \in H^1(C_t, \mathbb{Q})$ which is image of δ_i' via the diffeomorphism. By Lemma 3.2.19 the vanishing cohomology $H^1(C_t, \mathbb{Q})_{\mathsf{van}}$ is generated by these vanishing cycles δ_i, $i = 1, \ldots, M$, of the Lefschetz pencil L.

On the other hand, let $\tilde{\gamma}_i$ be the loop in V based at t such that $\tilde{\gamma}_i$ is equal to γ_i until t_i winds around the disk D_i once in the positive direction, and then returns to t via γ_i^{-1}. Recall that these loops $\tilde{\gamma}_i, i = 1, \ldots, M$, generate $\pi_1(V, t)$ and note that the image of the loops $\tilde{\gamma}_i$ via ρ_{f_V} are elements in $\mathsf{Aut}(H^1(C_t, \mathbb{Q})_{\mathsf{van}})$, that is,

$$\rho_{f_V}(\tilde{\gamma}_i) : H^1(C_t, \mathbb{Q})_{\mathsf{van}} \to H^1(C_t, \mathbb{Q})_{\mathsf{van}}$$

are automorphisms. Let $F \subset H^1(C_t, \mathbb{Q})_{\mathsf{van}}$ be a nontrivial vector subspace which is stable under the monodromy action ρ_{f_V}. We must prove that $F = H^1(C_t, \mathbb{Q})_{\mathsf{van}}$.

Let $0 \neq \alpha \in F$. Since by Proposition 3.2.21, $\langle , \rangle$ is non-degenerate on $H^1(C_t, \mathbb{Q})_{\mathsf{van}}$ there exists $i \in \{1, \ldots, M\}$ such that

$$\langle \alpha, \delta_i \rangle \neq 0.$$

By the Picard-Lefschetz formula (Theorem 3.4.2) for $\alpha \in F \subset H^1(C_t, \mathbb{Q})_{\mathsf{van}}$ one has

$$\rho_{f_V}(\tilde{\gamma}_i)(\alpha) = \alpha \pm \langle \alpha, \delta_i \rangle \delta_i,$$

or equivalently,

$$\rho_{f_V}(\tilde{\gamma}_i)(\alpha) - \alpha = \pm\langle \alpha, \delta_i \rangle \delta_i.$$

Since, by assumption, F is a vector subspace of $H^1(C_t, \mathbb{Q})_{\mathsf{van}}$ which is stable under the monodromy action ρ_{f_V} (so $\rho_{f_V}(\tilde{\gamma}_i)(\alpha) \in F$) we have

$$\rho_{f_V}(\tilde{\gamma}_i)(\alpha) - \alpha = \pm\langle \alpha, \delta_i \rangle \delta_i \in F.$$

Then

$$\delta_i \in F.$$

But, Corollary 3.4.11 shows that all the vanishing cycles are conjugate under the monodromy action, so F, which is stable under the monodromy action, must contain all the vanishing cycles. Thus

$$F = H^1(C_t, \mathbb{Q})_{\mathsf{van}}.$$

□

By Zariski's theorem 3.4.9 this implies that the global monodromy representation

$$\rho_{f_U} : \pi_1(U, t) \to \mathsf{Aut}(H^1(C_t, \mathbb{Q})_{\mathsf{van}})$$

is also irreducible (see Theorem 3.4.14).

On the other hand, by Lemma 2.3.6 (fact 3) we have $H^1(C_t, \mathbb{Z}) \overset{w_{t*}}{\simeq} H^1(J_t, \mathbb{Z})$ is an isomorphism, then since in particular C_t is compact we get the isomorphism

$$H^1(C_t, \mathbb{Q}) = H^1(C_t, \mathbb{Z}) \otimes \mathbb{Q} \overset{(w_{t*})_{\mathbb{Q}}}{\simeq} H^1(J_t, \mathbb{Z}) \otimes \mathbb{Q} = H^1(J_t, \mathbb{Q})$$

induced by w_{t*} (see [29, §7.1.1]).

Let $L_t = (w_{t*})^{-1}_{\mathbb{Q}}(H^1(A_t, \mathbb{Q}))$ be the (pre)image in $H^1(C_t, \mathbb{Q})$ of $H^1(A_t, \mathbb{Q}) \subset H^1(J_t, \mathbb{Q})$ under the isomorphism $(w_{t*})^{-1}_{\mathbb{Q}}$. Then L_t is a $\mathbb{Q}$-vector subspace in $H^1(C_t, \mathbb{Q})$.

Recall also that $H^1(B_t, \mathbb{Z}) \overset{w_{t*}}{\simeq} H^1(C_t, \mathbb{Z})_{\mathsf{van}}$ (see final part of the proof of item *a*)), then it follows that

$$H^1(B_t, \mathbb{Q}) \overset{(w_{t*})_{\mathbb{Q}}}{\simeq} H^1(C_t, \mathbb{Q})_{\mathsf{van}}.$$

Since in item a) we proved that $H^1(A_t, \mathbb{Z}) \subset H^1(B_t, \mathbb{Z})$ we get

$$H^1(A_t, \mathbb{Q}) \subset H^1(B_t, \mathbb{Q}),$$

this implies that

$$L_t \subset H^1(C_t, \mathbb{Q})_{\mathsf{van}}$$

in $H^1(C_t, \mathbb{Q})$ via the isomorphism $(w_{t*})_{\mathbb{Q}}$. Moreover, L_t is a vector subspace in $H^1(C_t, \mathbb{Q})_{\mathsf{van}}$ which has a Hodge structure on it since it corresponds to the abelian subvariety A_t of

J_t, so we assume that it is invariant under the monodromy representation ρ_{f_U} (see also Proposition 3.3.15). Then, since the monodromy action

$$\rho_{f_U} : \pi_1(U, t) \to \mathsf{Aut}(H^1(C_t, \mathbb{Q})_{\mathsf{van}})$$

on $H^1(C_t, \mathbb{Q})_{\mathsf{van}}$ is irreducible, either

- $L_t \overset{(w_{t*})_{\mathbb{Q}}}{\simeq} H^1(A_t, \mathbb{Q}) = 0$ and then $A_t = 0$, or
- $L_t \overset{(w_{t*})_{\mathbb{Q}}}{\simeq} H^1(A_t, \mathbb{Q}) = H^1(C_t, \mathbb{Q})_{\mathsf{van}} \overset{(w_{t*})_{\mathbb{Q}}}{\simeq} H^1(B_t, \mathbb{Q})$ and then $A_t = B_t$.

$\square$

Chapter 5

Applications

In this chapter we prove a result on 0-cycles on surfaces as an application of the theorem on the Gysin kernel (Theorem 4.1.1), and we study the connection of this result with Bloch's conjecture and constant cycles curves.

Bloch conjecture is the converse of a criterion given by Mumford to determine when the Chow groups of 0-cycles on surfaces are not representable or equivalently when the Chow groups of 0-cycles on surfaces are not finite dimensional. More precisely Bloch's conjecture states

Conjecture 5.0.1 (Bloch's conjecture). Let S be a smooth projective surface over $\mathbb{C}$. If $p_g(S) = 0$, then

$$alb_S : \mathsf{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S)$$

is an isomorphism.

Or equivalently (see Definition 1.3.25 or [30, Theorem 10.11]),

Conjecture 5.0.2 (Bloch's conjecture). Let S be a smooth projective surface over $\mathbb{C}$. If $p_g(S) = 0$, then $\mathsf{CH}_0(S)_{\deg=0}$ is representable.

Also equivalently (see Proposition 1.3.32 or [30, Proposition 10.10]),

Conjecture 5.0.3 (Bloch's conjecture). Let S be a smooth projective surface over $\mathbb{C}$. If $p_g(S) = 0$, then $\mathsf{CH}_0(S)_{\deg=0}$ is finite dimensional.

On the other hand, the notion of constant cycles curves was introduced by Huybrechts on $K3$ surfaces, see [16], but it can be defined for arbitrary surfaces. The most important examples of constant cycles curves are provided by rational curves, but not every constant cycle curve is rational so it is still not known how much weaker the notion of constant cycles curves really is.

As constant cycles curves of bounded order resemble rational curves in many ways, Huybrechts restated two conjectures on rational curves for constant cycles curves. These two conjectures for constant cycles curves due to Huybrechts are (see [16]):

Conjecture 5.0.4. For any $K3$ surface S there exists a positive integer $n > 0$ such that the union $\bigcup C \subset S$ of all constant cycles curves $C \subset S$ of order $\leq n$ is dense.

Conjecture 5.0.5. Let S be a complex $K3$ surface. Then any point $x \in S$ with $[x] = c_S$ is contained in a constant cycle curve.

5.1 A theorem on zero cycles on surfaces

Recall that $\mathrm{Alb}(S) = J^3(S)$ is the abelian variety corresponding or associated to the Hodge structure on $H^3(S,\mathbb{Z})$, and that for $t \in U$, $J_t = J^1(C_t)$ is the abelian variety corresponding or associated to the Hodge structure on $H^1(C_t,\mathbb{Z})$ and B_t is the abelian subvariety in J_t corresponding to the Hodge substructure on

$$H^1(C_t,\mathbb{Z})_{\mathsf{van}} = \mathsf{Ker}(H^1(C_t,\mathbb{Z}) \to H^3(S,\mathbb{Z})).$$

(see Definition 2.3.2):

By definition we have a short exact sequence of Hodge structures

$$0 \to H^1(C_t,\mathbb{Z})_{\mathsf{van}} \to H^1(C_t,\mathbb{Z}) \to H^3(S,\mathbb{Z}) \to 0$$

which give rise to a short exact sequence of abelian varieties

$$0 \to B_t \to J_t \to \mathrm{Alb}(S) \to 0,$$

this means that $B_t \to J_t$ is injective, $J_t \to \mathrm{Alb}(S)$ is surjective and

$$\mathrm{im}(B_t \to J_t) = \mathsf{Ker}(J_t \to \mathrm{Alb}(S)).$$

So

$$\mathrm{Alb}(S) \cong \frac{J_t}{B_t}.$$

As an application of Theorem 4.1.1 we have the following result

Theorem 5.1.1 (A theorem on the 0-cycles on surfaces)**.** *If*

$$alb_S : \mathsf{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S) \cong \frac{J_t}{B_t}$$

is not an isomorphism, for a very general $t \in U$, then G_t is countable.

Proof. We must prove that there exits an c-open in U such that for all t in this c-open, if $alb_S : \mathsf{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S) \cong \frac{J_t}{B_t}$ is not an isomorphism then G_t is countable.

Since by item a) of Theorem 4.1.1 we have that for every $t \in U$

$$G_t = \mathsf{Ker}(r_{t*}) = \bigcup_{\text{countable}} \text{translates of } A_t \text{ in } J_t$$

it is enough to prove that there exits an c-open in U such that for all t in this c-open, if $alb_S : \mathsf{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S) \cong \frac{J_t}{B_t}$ is not an isomorphism then $A_t = 0$.

By item b) of Theorem 4.1.1, we know that there exists an c-open U_0 such that for every $t \in U_0$ we have that $A_t = 0$ or for every $t \in U_0$ we have that $A_t = B_t$.

So it is enough to prove that for every $t \in U_0$, if $alb_S : \mathsf{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S) \cong \frac{J_t}{B_t}$ is not an isomorphism then $A_t = 0$.

Let $t \in U_0$ be any element of U_0. By contradiction, suppose that it is not true, i.e., that $alb_S : \mathsf{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S) \cong \frac{J_t}{B_t}$ is not an isomorphism and $A_t = B_t$. Then, we have that

$$G_t = \mathsf{Ker}(r_{t*}) = \bigcup_{\text{countable}} \text{translates of } B_t \text{ in } J_t$$

Now denote by

$$\pi_t : J_t = \mathsf{CH}_0(C_t)_{\deg=0} \to \mathrm{Alb}(S) \cong \frac{J_t}{B_t}$$

to the above morphism of abelian varieties induced by the homomorphism of Hodge structures $H^1(C_t, \mathbb{Z}) \to H^3(S, \mathbb{Z})$.

and consider

$$r_{t*} : J_t = \mathsf{CH}_0(C_t)_{\deg=0} \to \mathsf{CH}_0(S)_{\deg=0},$$

The Gysing homomorphism in Chow groups induced by the closed embedding r_t. Since $\pi_t : J_t \to \mathrm{Alb}(S)$ is surjective, for a fixed $z \in \mathrm{Alb}(S)$ there exits $x \in J_t$ such that $\pi_t(x) = z$. Then under $r_{t*} : J_t = \mathsf{CH}_0(C_t)_{\deg=0} \to \mathsf{CH}_0(S)_{\deg=0}$ we get $r_{t*}(x) \in \mathsf{CH}_0(S)_{\deg=0}$. So we get map

$$f_t : \mathrm{Alb}(S) \cong \frac{J_t}{B_t} \to \mathsf{CH}_0(S)_{\deg=0},$$

defined by $f_t(z) = r_{t*}(x)$.

Since, by assumption, $\mathsf{Ker}(\pi_t) \subset G_t = \ker(r_{t*})$ this map is well defined. Indeed, suppose that there are x_1, x_2 such that $\pi_t(x_1) = z$ and $\pi_t(x_2) = z$, then we have that $\pi_t(x_1) = \pi_t(x_2)$, then since π_t is an homomorphism we have $\pi_t(x_1 - x_2) = 0$, it follows that $x_1 - x_2 \in B_t$ and since, by our assumption, $B_t \subset \ker(r_{t*})$ we have that $r_{t*}(x_1 - x_2) = 0$, then $r_{t*}(x_1) = r_{t*}(x_2)$ because r_{t*} is a homomorphism. So, f_t is well defined since it does not depend on the choice of the preimage of z.

The composition

$$\mathsf{CH}_0(S)_{\deg=0} \stackrel{alb_S}{\to} \mathrm{Alb}(S) \stackrel{f_t}{\to} \mathsf{CH}_0(S)_{\deg=0}$$

is the identity. Indeed, since $\mathsf{CH}_0(S)_{\deg=0}$ is generated by points of the form $a - b$ where a, b lies in some C_t of our fixed family, it suffices to check that $a - b$ maps to $a - b$ under the composition. On the other hand, $alb_S : \mathsf{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S)$ is a surjection (see

[7, Introduction]), so $\mathsf{CH}_0(S)_{\deg=0}$ is isomorphic to $\mathrm{Alb}(S)$ which contradicts to the hypothesis, so we must have that $A_t = 0$, then G_t is countable and we are done. □

Corollary 5.1.2. *If S is a connected smooth projective surface over $\mathbb{C}$ with $p_g(S) \neq 0$, then G_t is countable for a very general $t \in U$.*

Proof. If $p_g(S) \neq 0$ by Mumford theorem we have that alb_S is not an isomorphism (see [30, Theorem 10.1]), so by Theorem 5.1.1 the Gysin kernel G_t is countable, for a very general $t \in U \subset \Sigma$. □

Example 5.1.3. Let S be a $K3$ surface or an abelian surface and Σ the complete linear system of a very ample divisor on S. Then $p_g(S) \neq 0$ (see [3, Chapter VIII]), so by Corollary 5.1.2 G_t is countable, for a very general $t \in U \subset \Sigma$.

5.2 Relation with Bloch's conjecture

In this section we show that the theorem on 0-cycles on surfaces gives us a criteria to prove Bloch's conjecture and that this theorem applied to surfaces with irregularity zero gives us a useful criteria to prove Bloch's conjecture to surfaces of general type.

The theorem on 0-cycles on surfaces (Theorem 5.1.1) is useful to prove Bloch's conjecture because its contrapositive form, and hence equivalent form is as follows

Corollary 5.2.1. *If G_t is uncountable, for a very general $t \in U$, then*

$$alb_S : \mathsf{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S)$$

is an isomorphism.

So, if a surface S, as in Theorem 4.1.1, has $p_g(S) = 0$ and we want to prove that Bloch's conjecture holds for this surface, i.e., alb_S is an isomorphism, it is enough to prove that G_t is not countable, for a very general $t \in U$ or that $A_t = B_t$ for a very general $t \in U$.

Bloch's conjecture has been proved for surfaces of special type i.e. for surfaces with Kodaira dimension less than 2, but for surfaces of general type, i.e., for surfaces with Kodaira dimension 2 it is not proved yet, except for some particular cases.

For surfaces of general type the relation between the theorem on 0-cycles of surfaces and Bloch's conjecture is expressed as follows.

Let S be a surface of general type with $p_g(S) = 0$. In this case $p_g(S) = 0$ implies that $q(S) = 0$, i.e., $\mathrm{Alb}(S) = 0$, and the Bloch's conjecture for surfaces of general type can be stated as follows

Conjecture 5.2.2 (Bloch's conjecture for surfaces of general type). Let S be a smooth projective surface over $\mathbb{C}$ of general type. If $p_g(S) = 0$, then

$$\mathsf{CH}_0(S)_{\deg=0} = 0,$$

i.e. any two closed points in S are rationally equivalent.

On the other hand, from Theorem 4.1.1 applied to surfaces with $q(S) = 0$ we get the following corollary

Corollary 5.2.3. *Let S, Δ_S, U, G_t, B_t and J_t be as Theorem 4.1.1. If in addition $q(S) = 0$. Then*

a) For every $t \in U$ there is an abelian variety $A_t \subset B_t$ such that

$$G_t = \bigcup_{countable} \textit{translates of } A_t$$

b) There exits a c-open subset $U_0 \subset U$ such that $A_t = 0$ for all $t \in U_0$ or $r_{t*} = 0$ for all $t \in U_0$.

c) Assume that Δ_S is an hypersurface. Then for every $t \in U$, $A_t = 0$ or $r_t = 0$.

Proof. For item a) there is nothing to prove.

By item b) of Theorem 4.1.1 there exists a c-open U_0 subset in U such that $A_t = 0$ for every $t \in U_0$ or $A_t = B_t$ for every $t \in U_0$.

In the first case, i.e., if $A_t = 0$ for every $t \in U_0$ we do not have anything to prove.

In the second case, i.e., if $A_t = B_t$ for every $t \in U_0$ then since by hypothesis $q(S) = 0$ we have $H^3(S, \mathbb{Z}) = 0$, so in particular for every $t \in U_0$ we have that $H^1(C_t, \mathbb{Z}) = H^1(C_t, \mathbb{Z})_{van}$, i.e., $B_t = J_t$. Then by item a) of Theorem 4.1.1 we have $G_t = J_t$ for every $t \in U_0$, i.e. $r_{t*} = 0$, for every $t \in U_0$ which proves item b) of this corollary.

By item c) of Theorem 4.1.1 one has that for any point $t \in U$, $A_t = 0$ or $A_t = B_t$.

Fix any arbitrary $t \in U$, then in the first case, i.e., if $A_t = 0$ we do not have anything to prove.

In the second case, i.e., if $A_t = B_t$ since by hypothesis $q(S) = 0$ we have $H^3(S, \mathbb{Z}) = 0$, so $H^1(C_t, \mathbb{Z}) = H^1(C_t, \mathbb{Z})_{van}$, i.e., $B_t = J_t$. Then by item a) of Theorem 4.1.1 we have $G_t = J_t$ i.e. $r_{t*} = 0$, which proves item c) of this corollary. □

Remark 5.2.4. In item b) and c) of the above corollary note that if $A_t = 0$ then it follows immediately that G_t is countable.

Remark 5.2.5. Note that item b) of Theorem 4.1.1 more precisely states that there exits a c-open subset $U_0 \subset U$ such that either $A_{\overline{\eta}} = 0$ in which case $A_t = 0$, for every $t \in U_0$, or $A_{\overline{\eta}} = B_{\overline{\eta}}$ in which case $A_t = B_t$, for every $t \in U_0$.

By Remark 4.2.3 it is enough to prove item b) at the level of (closed) very general points in U, that is, points $t \in U_0$ instead of proving item b) for the geometric generic point $\overline{\eta}$ of $\Sigma = \mathbb{P}^{d*}$.

From Theorem 5.1.1 on 0-cycles of surfaces applied to surfaces S with $q(S) = 0$ we get the following result

Corollary 5.2.6. *Assume in addition that S is a surface with $q(S) = 0$. If*

$$\mathsf{CH}_0(S)_{\deg=0} \neq 0$$

then for a very general $t \in U$, G_t is countable.

Proof. It is obvious since when $q(S) = 0$ then $\mathrm{Alb}(S) = 0$ then by Theorem 5.1.1 we are done. □

The above corollary is equivalent to the following

Corollary 5.2.7. *Assume in addition that S is a surface with $q(S) = 0$. If $r_{t*} = 0$, for a very general $t \in U$, then $\mathsf{CH}_0(S)_{\deg=0} = 0$.*

Proof. The contrapositive form of Corollary 5.2.6 tells us that if it is not true that G_t is countable, i.e., that $A_t = 0$, for a very general $t \in U$, then $\mathsf{CH}_0(S)_{\deg=0} = 0$. Then applying item b) of Corollary 5.2.3 we are done. □

So, if a surface S as in Theorem 4.1.1 is of general type with $p_g(S) = 0$ and we want to prove that Bloch's conjecture holds for this surface, i.e., $\mathsf{CH}_0(S)_{\deg=0} = 0$ it is enough to prove that $r_{t*} = 0$, for a very general $t \in U$ i.e. more precisely we have

Corollary 5.2.8. *Assume in addition that S is a surface of general type with $p_g(S) = 0$. Then $r_{t*} = 0$, for a very general $t \in U$, if and only if Bloch's conjecture holds for S.*

Proof. Since S is a surface of general type with $p_g(S) = 0$, it follows that $q(S) = 0$, then we apply Corollary 5.2.7 and we are done.

Reciprocally, assume that Bloch's conjecture holds for S then by Proposition 4.1 in [16] $r_{t*} = 0$ for every $t \in \Sigma = \mathbb{P}^{d*}$, then $r_{t*} = 0$, for a very general $t \in U$. □

5.3 Relation with constant cycle curves

In this section we show that the theorem on the Gysin kernel applied to surfaces with irregularity zero gives a criteria to determine when smooth curves in a linear system are constant cycles curves, we will also see that the theorem on 0-cycles on surfaces applied to surfaces with $q(S) = 0$ gives us a criteria to prove Bloch's conjecture using the notion of constant cycles curves.

Let S be a surface over $\mathbb{C}$.

Definition 5.3.1 (Pointwise constant cycle curve)**.** A curve $C \subset S$ is a *pointwise constant cycle curve* if all closed points $x \in C$ define the same class $[x] \in \mathsf{CH}^2(S)$.

Since we are working over $\mathbb{C}$ which is algebraically closed, we have the following equivalent definition (see [16, §3]).

Definition 5.3.2 (Pointwise constant cycle curve)**.** A curve $C \subset S$ is a *pointwise constant cycle curve* if and only if

$$r_{C*} : \mathrm{Pic}^0(\tilde{C}) = \mathsf{CH}_0(\tilde{C})_{\deg=0} \to \mathsf{CH}_0(S)$$

is the zero map. Here, $r_C : \tilde{C} \to S$ is the composition of the normalization $\tilde{C} \to C$ with the closed embedding $C \hookrightarrow S$.

By Lemma 3.8 in [16], we can define the notion of a constant cycle curve on any surface as follows

Definition 5.3.3 (Constant cycle curve)**.** Let S be a surface over $\mathbb{C}$. An integral curve $C \subset S$ is a *constant cycle curve* if and only if there exists a positive integer n such that

$$n \cdot [\eta_C] \in \mathrm{im}\Big(\mathsf{CH}^2(S) \to \mathsf{CH}^2\big(S \times_{\mathbb{C}} \mathbb{C}(\eta_C)\big)\Big),$$

where the generic point $\eta_C \in C$ is viewed as a closed point in $S \times_{\mathbb{C}} \mathbb{C}(\eta_C)$. Equivalently, $C \subset S$ is a constant cycle curve when

$$[\eta_C] \in \mathrm{im}\Big(\mathsf{CH}^2(S) \to \mathsf{CH}^2\big(S \times_{\mathbb{C}} \overline{\mathbb{C}(\eta_C)}\big)\Big),$$

if η_C is viewed as a point in the geometric generic fibre $S \times_{\mathbb{C}} \overline{\mathbb{C}(\eta_C)}$.

Definition 5.3.4 (Constant cycle curve)**.** Let S be a surface over $\mathbb{C}$. We call an arbitrary curve $C \subset S$ a *constant cycle curve* if every integral component of C is a constant cycle curve.

When the ground field is $\mathbb{C}$ these two notions coincide thanks to the following proposition, see Proposition 3.7 in [16].

Proposition 5.3.5. *Let S be a surface over an algebraically closed field k. Then a constant cycle curve $C \subset S$ is also a pointwise constant cycle curve. If k is uncountable, the converse holds true as well.*

Using the definition of constant cycle curves the Corollary 5.2.3 can be restated as follows

Corollary 5.3.6. *Let S, Δ_S, U, G_t, B_t and J_t be as Corollary 5.2.3. If in addition $q(S) = 0$. Then*

a) For every $t \in U$ there is an abelian variety $A_t \subset B_t$ such that

$$G_t = \bigcup_{countable} \textit{translates of } A_t$$

b) There exits a c-open subset $U_0 \subset U$ such that $A_t = 0$ for all $t \in U_0$, or C_t is a constant cycle curve for all $t \in U_0$.

c) Assume that Δ_S is an hypersurface. Then for every $t \in U$, $A_t = 0$ or C_t is a constant cycle curve.

Proof. Recall that since the ground field is $\mathbb{C}$ a smooth curve C_t is a constant cycle curve if $r_{t*} = 0$, and then apply Corollary 5.2.3. $\square$

So item c) of the corollary above tells us that the study of the Gysin kernel gives us a criteria to determine when a smooth curve in a linear system of a connected smooth projective surface S with $q(S) = 0$ is a constant cycle curve.

The following result shows the relation of the theorem on the 0-cycles on surfaces, i.e., Theorem 5.1.1, the notion of constant cycle curves and Bloch's conjecture.

Corollary 5.3.7. *Assume in addition that S is a surface of general type with $p_g(S) = 0$. Then the curve C_t is a constant cycle curve, for a very general $t \in U$, if and only if S satisfies Bloch's conjecture.*

Proof. We apply Corollary 5.2.8 and definition of constant cycle curve and we are done. $\square$

The above corollary says that if a surface S as in Theorem 4.1.1 is of general type with $p_g(S) = 0$ and we want to prove that Bloch's conjecture holds for this surface, i.e., $\mathsf{CH}_0(S)_{\deg=0} = 0$ it is enough to prove that C_t is a constant cycle curve, for a very general $t \in U$.

In the following examples let Σ be a complete linear system corresponding to a very ample divisor on the surface S.

Example 5.3.8. Let S be the minimal model of a product-quotient surface with $p_g(S) = 0$, then by Theorem 3.4. in [2] it is a surface of general type satisfying Bloch's conjecture; therefore by Corollary 5.3.7 then C_t is a constant cycle curve, for a very general $t \in U \subset \Sigma$.

Example 5.3.9. Let S be a surface as in [2, Corollary 3.5.], then by Corollary 5.3.7 C_t is a constant cycle curve, for a very general $t \in U \subset \Sigma$.

Example 5.3.10. Let S be a Catanese surface or a Barlow surface, then it is a surface of general type satisfying Bloch's conjecture (see [31]). So by Corollary 5.3.7 we have that C_t is a constant cycle curve, for a very general $t \in U \subset \Sigma$.

Example 5.3.11. Let S be a numerical Campedelli surface, i.e., a minimal surface S of general type with $p_g(S) = 0$, then it satisfies Bloch's conjecture (see [19]). It follows by Corollary 5.3.7 that C_t is a constant cycle curve, for a very general $t \in U \subset \Sigma$.

Example 5.3.12. Let S be a surface as in [25], since they are surfaces of general type with $p_g(S) = 0$ satisfying Bloch's conjecture, by Corollary 5.3.7 we have that C_t is a constant cycle curve, for a very general $t \in U \subset \Sigma$.

Appendix A

A.1 Future Research Topics

A.1.1 The Gysin kernel

Let S be a smooth projective and connected surface over $\mathbb{C}$ with $\dim(\Delta_S) \neq d-1$. An interesting question is what happens to the Gysin kernel G_t or more precisely to A_t when t is outside of U_0, i.e., in $U \setminus U_0$ which is called by Guletskii the ***misty locus***, in which no uniform behaviour is expected and it is not discussed in the present work. One interesting future work is to follow a concrete approach to understand the misty locus, it can be done working out particular examples for different types of surfaces which can provide more intuition to determine in which cases $A_t = 0$ and then the Gysin kernel is countable, in which cases $A_t = B_t$ and then the Gysin kernel is a countable union of translates of B_t and in which cases $0 \neq A_t \neq B_t$. Therefore providing information about the Gysin kernel G_t.

A.1.2 The Gysin kernel over an arbitrary algebraically closed field

Since the study of the Gysin kernel on surfaces with irregularity zero gives a criterion to determine constant cycle curves (see item c) and also item b) of Corollary 5.3.6), another interesting direction is to study whether we can get the analogous of the Theorem on the Gysin kernel over an arbitrary algebraically closed field.

This result may help to answer the following question due to Huybrechts (see [16, Introduction])

Question. Is it true that for all K3 surfaces over $\overline{\mathbb{F}}_q$ every curve is a constant cycle curve?

It is important to note that the above question is related to the following conjecture for $K3$ surfaces over global fields

Conjecture A.1.1. (Bloch-Beilinson) If X be a K3 surface over $\overline{\mathbb{F}}_p$, then

$$\mathsf{CH}^2(X \times_{\overline{\mathbb{F}}_p} \overline{\mathbb{F}_p(t)}) \simeq \mathbb{Z}.$$

Another interesting question is the following (see [16, Introduction]).

Question. Let Let X be a K3 surface over $\overline{\mathbb{F}}_p$. Is it true that every closed point $x \in X$ is contained is a constant cycle curve?

The answer to this question can help to prove the above Bloch-Beilinson conjecture.

A.1.3 Constant cycle curves

The notion of constant cycle curves originated when Beauville and Voisin in [4] described a distinguished element of the Chow group of 0-cycles $\mathsf{CH}_0(X)$ when X is a K3 surface over $\mathbb{C}$ or over $\overline{\mathbb{Q}}$. This distinguished element is denoted by c_X and satisfies the following properties: $c_2(X) = 24 \cdot c_X$ and $c_1(L)^2 \in \mathbb{Z} \cdot c_X$, where L is a line bundle on X.

This distinguished class is also interesting from an arithmetic point of view as is stated in the following conjecture

Conjecture A.1.2. (Bloch-Beilinson) If X is a K3 surface over $\overline{\mathbb{Q}}$ and $x \in X(\overline{\mathbb{Q}})$, i.e., x is a $\overline{\mathbb{Q}}$-rational point, then $[x] = c_X$.

Beauville and Voisin in [4] proved that if $x \in C$ with C rational, then $[x] = c_X$. This property inspired the Definition 5.3.1 of a pointwise constant cycle curve. If the field k is algebraically closed the above definition is equivalent to

Definition A.1.3. Let X be a projective K3 surface over a field k. A curve C in X is a *pointwise constant cycle curve* if for all closed points $x \in C$ we have $[x] = c_X \in \mathsf{CH}_0(X)$.

This last definition relates the study pointwise constant cycle curves, Chow groups, the distinguished cycle c_X with the study of the Gysin kernel because the above definition over an algebraically closed field k is equivalent to Definition 5.3.2 which in terms of the Gysin kernel can be restated as follows

Definition A.1.4. A curve $C \subset X$ is a *pointwise constant cycle curve* if and only if

$$G_C = \mathsf{Ker}(r_{C*} : \mathrm{Pic}^0(\tilde{C}) = \mathsf{CH}_0(\tilde{C})_{\deg=0} \to \mathsf{CH}_0(X)) = \mathrm{Pic}^0(\tilde{C}),$$

that is, the Gysin kernel of the resolution of singularities of the curve C coincides with $\mathrm{Pic}^0(\tilde{C}) = J(\tilde{C})$.

When the ground field is $\mathbb{C}$ or more generally over an uncountable algebraically closed field the notion of pointwise constant cycle curve coincides with the notion of constant cycle curves (see Definition 5.3.2) thanks to Proposition 3.7 in [16]. So, the study of the Gysin kernel is also related to the study of constant cycle curve in this case.

There are also interesting conjectures for constant cycles curves as the following which is stated by D. Huybrechts.

Conjecture A.1.5. (D. Huybrechts) For every $K3$ surface over an algebraically closed field of characteristic zero, there exist an $n > 0$ such that the union of all constant cycle curves C of X of order $\leq n$ is dense.

A.1.4 Bloch's conjecture

A complete understanding of the Gysin kernel can be used to proved Bloch's conjecture for many particular cases. This was pointed out to us by Guletskii whose approach to prove Bloch's conjecture for surfaces with involution is as follows:

- Let S be a smooth projective surface over an algebraically closed field k of characteristic 0, equipped with a regular involution ι, i.e., with an action of the order two (cyclic) group $G = \{id, \iota\}$ generated by ι.
- Let $\frac{S}{\iota}$ or $\frac{S}{G}$ be the quotient surface of orbits of S under the action of G.
- Let $\widetilde{(\frac{S}{\iota})}$ be the resolution of the quotient surface $\frac{S}{\iota}$.
- Let $q(S) = h^{0,1} = \dim(H^1(S, \mathcal{O}_S))$ be the irregularity of S.
- Let $M(S) = (S, \Delta_S, 0)$ be the motive of S. Here Δ_S is the fundamental class associated to the image of the diagonal morphism $\Delta : S \to S \times S$, and $0 \in \mathbb{Z}$.

A complete understanding of the Gysin kernel would help to prove the following claim.

Claim A.1.6. Assume that $q(S) = 0$. If $\mathsf{CH}_0(\widetilde{(\frac{S}{\iota})})_{\deg=0} = 0$ then $M(S)$ is finite dimensional.

Then, observing that for surfaces with $p_g(S) = 0$ the following facts are true:

- $M(S)$ is finite dimensional if and only if Bloch's conjecture holds for S.
- Assume in addition that S is of general type. Then, $\mathsf{CH}_0(\widetilde{(\frac{S}{\iota})})_{\deg=0} = 0$ if and only if $M(\widetilde{(\frac{S}{\iota})})$ is finite dimensional.

One can prove the following claim

Claim A.1.7. Let S be of general type and $p_g(S) = 0$. Then Bloch's conjecture holds for S if and only if Bloch's conjecture holds for $\widetilde{(\frac{S}{\iota})}$.

Using this claim we can prove Bloch's conjecture for all numerical Godeaux surfaces with involutions, a "half" of Campedelli surfaces with involutions, the surface of Craighero and Gattazzo, some Catanese surfaces and other examples.

Bibliography

[1] Kalyan Banerjee and Vladimir Guletskiĭ. *Etale monodromy and rational equivalence for 1-cycles on cubic hypersurfaces in* $\mathbb{P}^5$. Mat. Sb., 211(2):3–45, 2020.

[2] Ingrid Bauer, Fabrizio Catanese, and Roberto Pignatelli. *Surfaces of general type with geometric genus zero: a survey.* In Complex and Differential Geometry: Conference held at Leibniz Universität Hannover, September 14–18, 2009, pages 1–48. Springer, 2011.

[3] Arnaud Beauville. *Complex algebraic surfaces.* London Math. Soc. Lecture Notes, 1996.

[4] Arnaud Beauville and Claire Voisin. *On the Chow ring of a K3 surface.* Journal of Algebraic Geometry, 13(3):417–426, 2004.

[5] Spencer Bloch. K_2 *of Artinian Q-algebras, with application to algebraic cycles.* Comm. Algebra, 3:405–428, 1975.

[6] Spencer Bloch. *An example in the theory of algebraic cycles.* Lecture Notes in Mathematics, 551:1–29, 1976.

[7] Spencer Bloch. *Lectures on algebraic cycles.* Series IV, Durham, NC, 16:xxiv+130, 2010.

[8] Spencer Bloch, Arnold Kas, and David Lieberman. *Zero cycles on surfaces with* $p_g = 0$. Compositio Math., 33(2):135–145, 1976.

[9] David Eisenbud and Joe Harris. *3264 & All That Intersection Theory in Algebraic Geometry*, 2013.

[10] William Fulton. *Intersection theory.* Princeton University Press, 2016.

[11] William Fulton and Johan Hansen. *A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings.* Ann. of Math. (2), 110(1):159–166, 1979.

[12] Evgeny Goncharov. *Weil Conjectures I (translation of La Conjecture de Weil I by Pierre Deligne).* arXiv e-prints, pages arXiv–1807, 2018.

[13] Joe Harris. *Algebraic geometry: a first course*, volume 133. Springer Science & Business Media, 2013.

[14] Robin Hartshorne. *Equivalence relations on algebraic cycles and subvarieties of small codimension.* In Proc. Symp. Pure Math, volume 29, pages 129–164, 1975.

[15] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 2013.

[16] Daniel Huybrechts. *Curves and cycles on K3 surfaces.* Foundation Compositio Mathematica, 2014.

[17] Steven Kleiman. *Algebraic cycles and the Weil conjectures*, volume 3 of Adv. Stud. Pure Math. North-Holland, Amsterdam, 1968.

[18] Klaus Lamotke. The topology of complex projective varieties after s. lefschetz. *Topology*, 20(1):15–51, 1981.

[19] Robert Laterveer. *Bloch's conjecture for some numerical Campedelli surfaces.* Asian Journal of Mathematics, 25(1):49–64, 2021.

[20] James Milne. *Abelian Varieties (v2.00)*, 2008. Available at www.jmilne.org/math/.

[21] James Milne. *Lectures on Etale Cohomology (v2.21)*, 2013. Available at www.jmilne.org/math/.

[22] David Mumford. *Rational equivalence of 0-cycles on surfaces.* Journal of mathematics of Kyoto University, 9(2):195–204, 1969.

[23] Jacob Murre. *Un résultat en théorie des cycles algébriques de codimension deux.* C. R. Acad. Sci. Paris Sér. I Math., 296(23):981–984, 1983.

[24] Jacob Murre, Jan Nagel, and Chris Peters. *Lectures on the theory of pure motives*, volume 61 of University Lecture Series. American Mathematical Society, Providence, RI, 2013.

[25] Claudio Pedrini and Charles Weibel. *Some surfaces of general type for which Bloch's conjecture holds.* Recent Advances in Hodge Theory (Ed. M. Kerr), London Mathematical Society Lecture Note Series, 427:308–329, 2013.

[26] Chris Peters and Joseph Steenbrink. *Mixed hodge structures*, volume 52. Springer Science & Business Media, 2008.

[27] A. A. Roĭtman. *Rational equivalence of zero-dimensional cycles.* Mat. Sb. (N.S.), 89(131):569–585, 671, 1972.

[28] Evgueni Tevelev. *Projective Duality and Homogeneous Spaces: Invariant Theory and Algebraic Transformation Groups, IV*. Encyclopaedia Math. Sci, 133, 2005.

[29] Claire Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2002. Translated from the French original by Leila Schneps.

[30] Claire Voisin. *Hodge theory and complex algebraic geometry. II*, volume 77 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2003. Translated from the French by Leila Schneps.

[31] Claire Voisin. *Bloch's conjecture for Catanese and Barlow surfaces*. Journal of Differential Geometry, 97(1):149–175, 2014.

Department of Mathematical Sciences, Universidad Nacional de Ingenería-Instituto de Matemáticas y Ciencias Afines, Lima, Perú